Calculus Name________________________

Optimization

Solve each optimization problem.

1) A farmer wants to construct a rectangular pigpen using 200 ft of fencing. The pen will be built next to an existing stone wall, so only three sides of fencing need to be constructed to enclose the pen. What dimensions should the farmer use to construct the pen with the largest possible area?

2) A rancher wants to construct two identical rectangular corrals using 100 ft of fencing. The rancher decides to build them adjacent to each other, so they share fencing on one side. What dimensions should the rancher use to construct each corral so that together, they will enclose the largest possible area?

3) A supermarket employee wants to construct an open-top box from a 16 by 30 in piece of cardboard. To do this, the employee plans to cut out squares of equal size from the four corners so the four sides can be bent upwards. What size should the squares be in order to create a box with the largest possible volume?

4) Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 108 ft^3 of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass?

5) A graphic designer is asked to create a movie poster with a 98 in^2 photo surrounded by a 4 in border at the top and bottom and a 2 in border on each side. What overall dimensions for the poster should the designer choose to use the least amount of paper?

6) Which points on the graph of $y = 6 - x^2$ are closest to the point $(0, 3)$?

7) Which point on the graph of $y = \sqrt{x}$ is closest to the point $(2, 0)$?

8) An architect is designing a composite window by attaching a semicircular window on top of a rectangular window, so the diameter of the top window is equal to and aligned with the width of the bottom window. If the architect wants the perimeter of the composite window to be 10 ft, what dimensions should the bottom window be in order to create the composite window with the largest area?

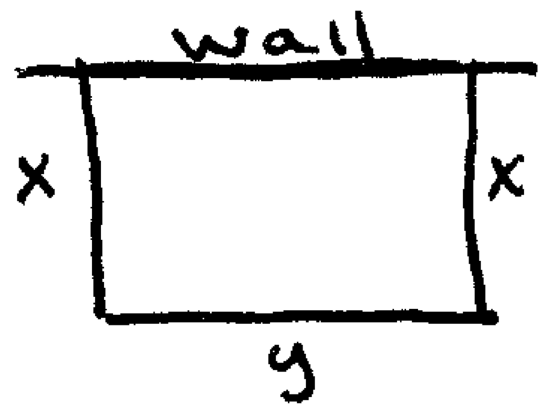
9) Two vertical poles, one 4 ft high and the other 8 ft high, stand 9 feet apart on a flat field. A worker wants to support both poles by running rope from the ground to the top of each post. If the worker wants to stake both ropes in the ground at the same point, where should the stake be placed to use the least amount of rope?

10) A geometry student wants to draw a rectangle inscribed in the ellipse $x^2 + 4y^2 = 16$. What is the area of the largest rectangle that the student can draw?

11) A geometry student wants to draw a rectangle inscribed in a semicircle of radius 6. If one side must be on the semicircle's diameter, what is the area of the largest rectangle that the student can draw?

Optimization Worksheet #2 Answers

1. Maximize Area Rectangle : LW
   Constraint: 200 feet fencing for 3 sides



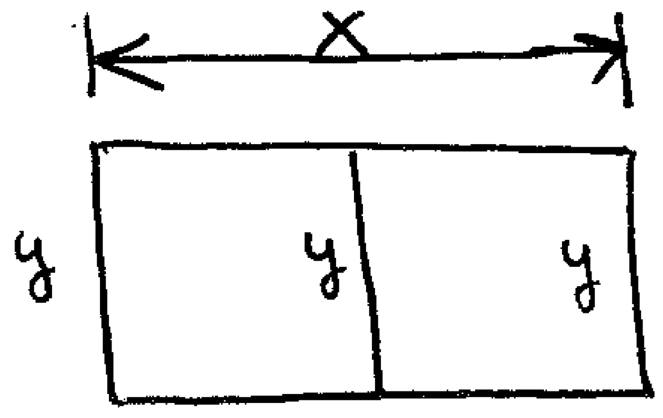
$$2x+y=200 \rightarrow y=200-2x$$

$$A = LW$$
$$A = xy$$
$$A = x(200-2x)$$
$$A = 200x - 2x^2$$
$$A' = 200 - 4x = 0$$

$\boxed{x = 50' \text{ and } y = 100'}$

$\boxed{\text{The dimensions are } 100' \times 50'.}$

2, Maximize Area of 2 pens : LW
   Constraint : 100 feet fencing



$$2x + 3y = 100$$
$$y = \frac{100}{3} - \frac{2}{3}x$$

$$A = LW$$
$$A = xy$$
$$A = x\left(\frac{100}{3} - \frac{2}{3}x\right)$$
$$A = \frac{100}{3}x - \frac{2}{3}x^2$$
$$A' = \frac{100}{3} - \frac{4}{3}x = 0$$
$$\frac{100}{3} = \frac{4}{3}x$$

$\boxed{x = 25 \qquad y = \frac{100}{3} - \frac{2}{3}(25) = \frac{50}{3}}$

$\boxed{\text{Each corral should be } \frac{25}{2}' \times \frac{50}{3}'.}$

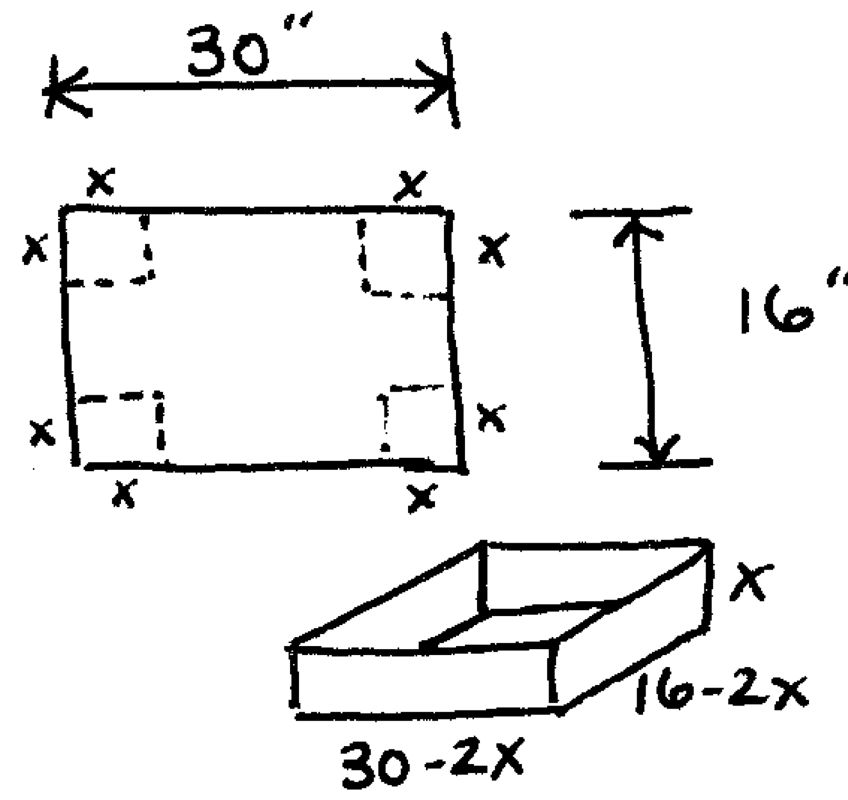
3. Maximize Volume = LWH

Constraints: Cardboard = 16" x 30"

$L = 30 - 2x$

$W = 16 - 2x$

$H = x$



$V = LWH$

$V = (30-2x)(16-2x)(x)$

$V = 4x^3 - 92x^2 + 480x$

$V' = 12x^2 - 184x + 480 = 0$

$= 4(3x-10)(x-12) = 0$

$x = {}^{10}/_{3}$ or $\boxed{\cancel{12}}$ ← cannot equal 12, because $W = 16 - 2x$. ← $\boxed{x < 8!}$

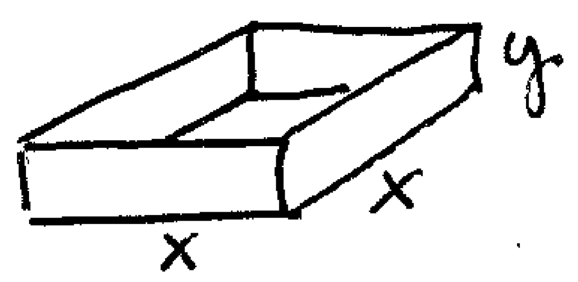
$\boxed{\text{The squares should be } {}^{10}/_{3}\text{" square}}$

4. Minimize Surface Area $= x^2 + 4xy$

Constraint $V = 108 \text{ ft}^3$

$V = x^2 y = 108$

$y = \frac{108}{x^2}$



$SA = x^2 + 4xy$

$SA = x^2 + 4x\left(\frac{108}{x^2}\right)$

$SA = x^2 + 432x^{-1}$

$SA' = 2x - \frac{432}{x^2} = 0$

$2x = \frac{432}{x^2}$

$x = \sqrt[3]{216} = 6; \quad y = \frac{108}{36} = 3$

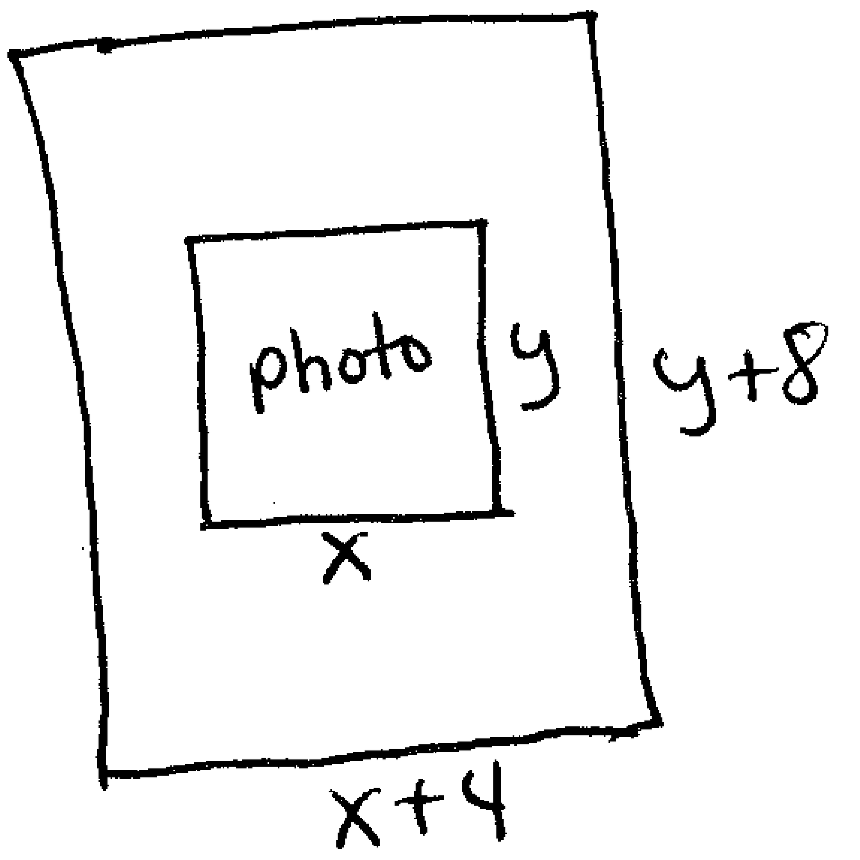
$\boxed{\text{The dimensions of the aquarium should be } 6' \times 6' \times 3'}$

5. One way of doing it:

Minimize Area $= (x+4)(y+8)$

Constraint = Photo Area $= 98$

$$xy = 98$$
$$y = \frac{98}{x}$$



$$A = (x+4)(y+8)$$
$$A = (x+4)\left(\frac{98}{x} + 8\right)$$
$$A = 98 + 8x + 392x^{-1} + 32$$
$$A = 8x + 392x^{-1} + 130$$
$$A' = 8 - \frac{392}{x^2} = 0$$
$$8x^2 = 392$$
$$x = 7'' \quad y = 14''$$

Length $= x+4 = 11$

Width $= y+8 = 22$

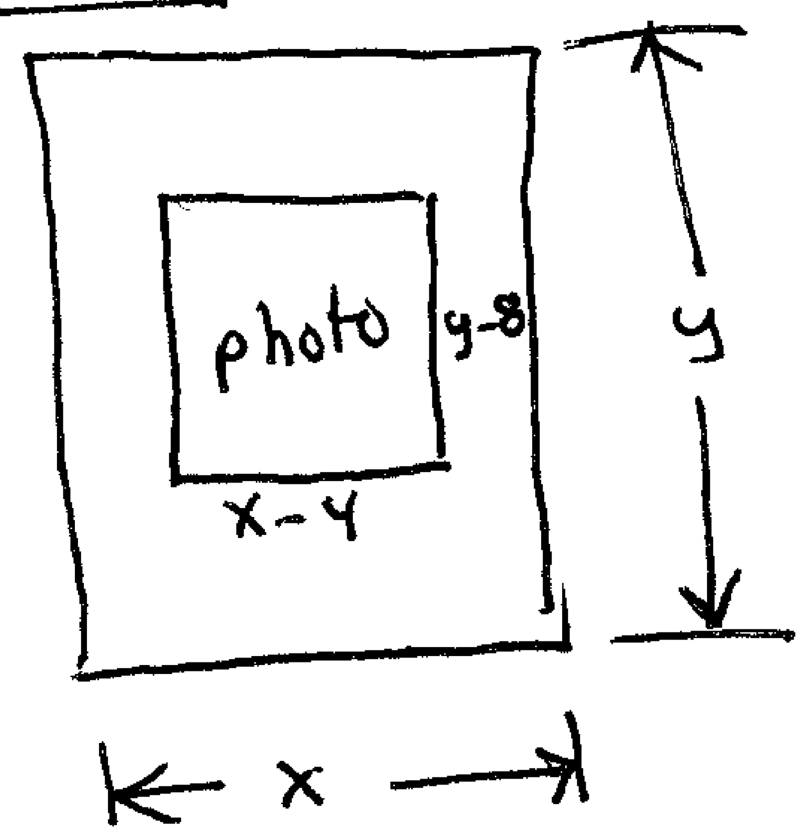
$\boxed{\text{Overall dimensions of poster are } 11'' \times 22''.}$

A different way:

Minimize Area $= xy$

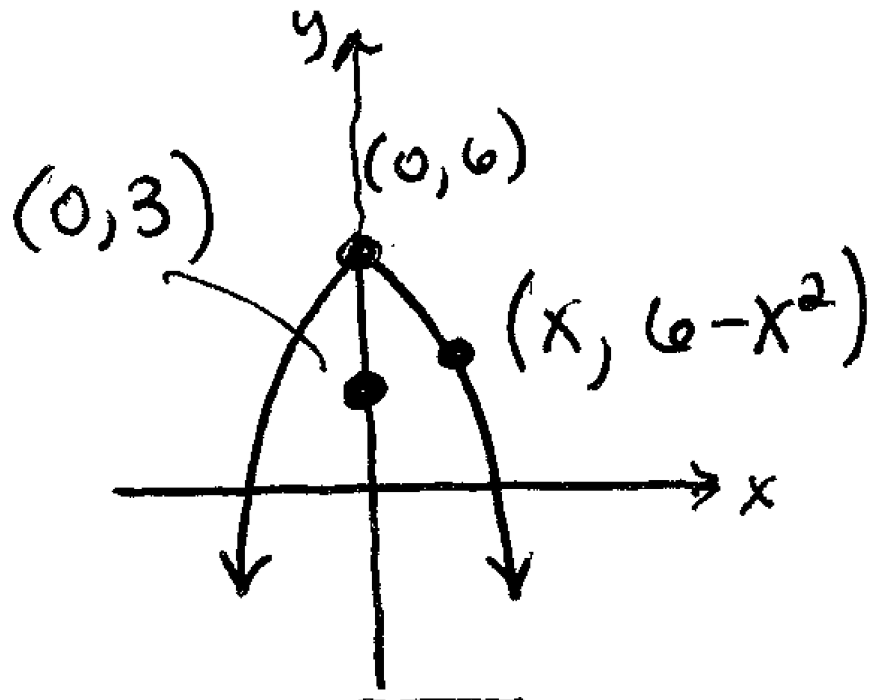
Constraint = Photo Area $= 98$

$$(x-4)(y-8) = 98$$
$$y = \frac{98}{x-4} + 8$$



$$A = (x)\left(\frac{98}{x-4} + 8\right)$$
$$A = \frac{98x}{x-4} + 8x$$
$$A' = \frac{(x-4)(98) - 98x}{(x-4)^2} + 8 = \frac{98x - 392 - 98x}{(x-4)^2} + 8$$
$$= \frac{-392}{(x-4)^2} + 8 = 0$$
$$\frac{392}{8} = (x-4)^2 \qquad \boxed{x = 11''; \ y = 22''}$$

6. Minimize Distance to $(0,3)$
Constraint $y = 6 - x^2$



Distance Formula: $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The points are $(0,3)$ and $(x, 6-x^2)$

$$D = \sqrt{(6-x^2-3)^2 + (x-0)^2}$$

$$D = (x^4 - 5x^2 + 9)^{1/2}$$

$$D' = \tfrac{1}{2}(x^4 - 5x^2 + 9)^{-1/2}(4x^3 - 10x) = 0$$

$$4x^3 - 10x = 0 \qquad \leftarrow \text{ the denominator cannot } = 0\text{!}$$

$$2x(2x^2 - 5) = 0$$

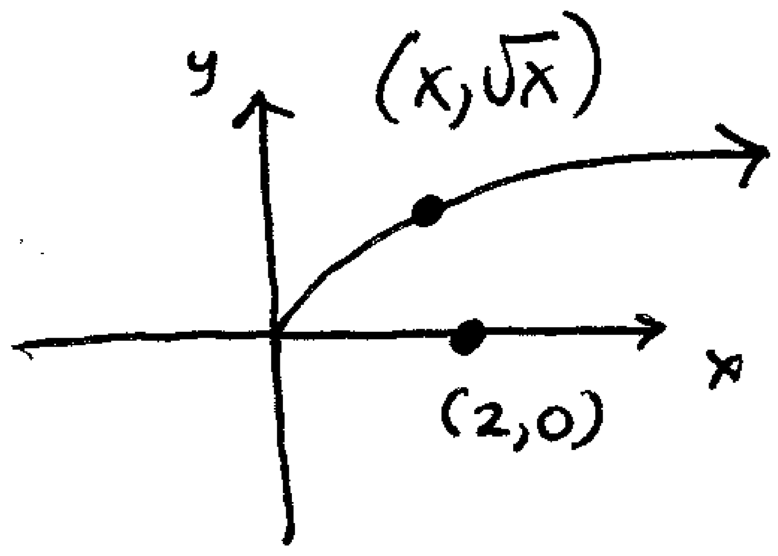
$$x = 0, \pm\sqrt{5/2}$$

Check answers: If $x = 0$, $y = 6$! Our original point.
$x = 0$ is an extranneous solution.

If $x = \pm\sqrt{5/2}$, $y = 7/2$, $D = \pm 1.66$ ✓

$\boxed{(\pm\sqrt{5/2},\ 7/2) \text{ are the points closest to } (0,6).}$

7. Minimize Distance from (2,0) to $y=\sqrt{x}$.
Constraint $y=\sqrt{x}$



$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Points are $(2,0)$ and $(x,\sqrt{x})$

$$D=\sqrt{(2-x)^2+(0-\sqrt{x})^2}$$

$$D=\sqrt{x^2-4x+4+x}$$
$$D=(x^2-3x+4)^{1/2}$$

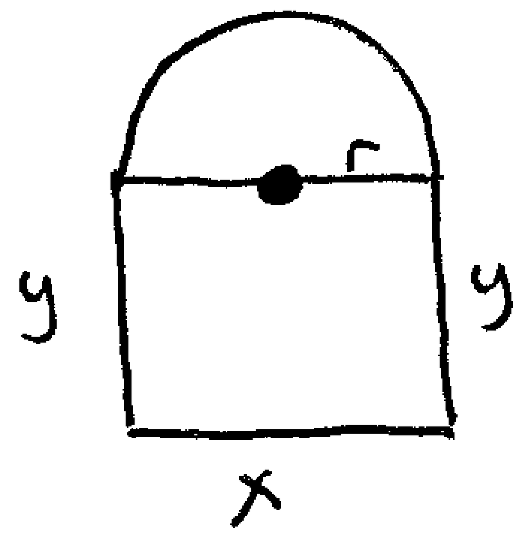
$$D'=\tfrac{1}{2}(x^2-3x+4)^{-1/2}(2x-3)$$

$2x-3=0$
$x=3/2 ; \; y=\frac{\sqrt{6}}{2}$

$\boxed{\text{The point } (3/2, \sqrt{6}/2) \text{ is the point closest to } (2,0).}$

8. Maximize Area of rectangular bottom window. $A = LW = xy$
Constraint: Perimeter of composite window $= 10'$

$$P = 10 = x + 2y + \tfrac{1}{2}(2\pi r)$$



and $r = \tfrac{1}{2}x$

so substitute r and solve for y

$$10 = x + 2y + \tfrac{1}{2}\left(2\pi\left(\tfrac{1}{2}x\right)\right)$$

$$y = \tfrac{1}{2}\left(10 - x - \tfrac{\pi}{2}x\right)$$
$$y = 5 - \tfrac{1}{2}x - \tfrac{\pi}{4}x)$$

Want to create composite window with largest area.

$$A = xy + \tfrac{1}{2}(\pi r^2)$$
$$A = x\left(5 - \tfrac{1}{2}x - \tfrac{\pi}{4}x\right) + \tfrac{1}{2}\left(\pi\left(\tfrac{1}{2}x\right)^2\right)$$
$$A = 5x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2$$
$$A = 5x - x^2\left(\tfrac{1}{2} + \tfrac{\pi}{4} - \tfrac{\pi}{8}\right)$$
$$A = 5x - x^2\left(\tfrac{4+\pi}{8}\right)$$

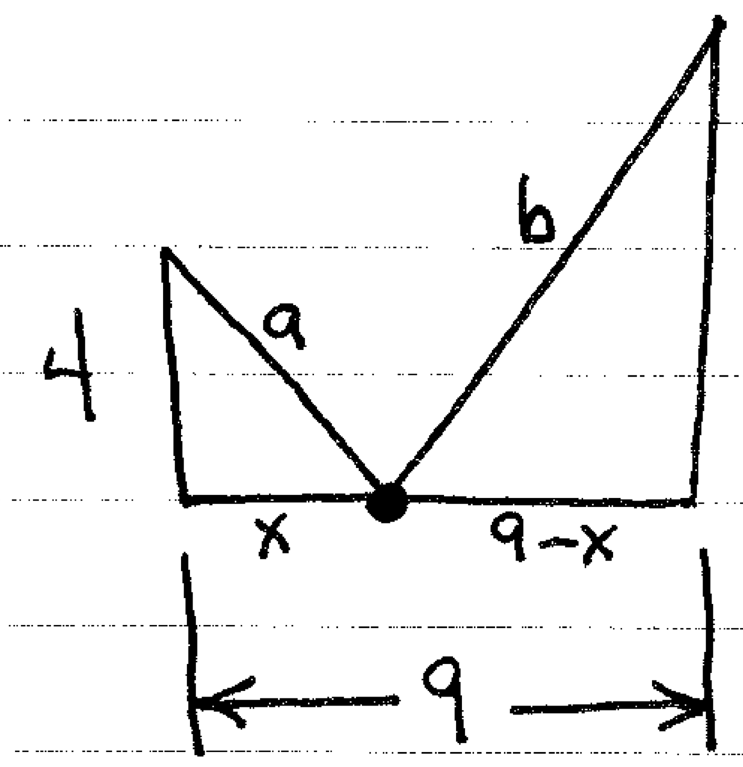
$$A' = 5 - \frac{4+\pi}{4}x = 0$$

$$5 = \frac{4+\pi}{4}x$$

$\boxed{x = \frac{20}{4+\pi} \text{ feet}}$ $\boxed{y = \frac{10}{4+\pi} \text{ feet}}$

$\boxed{\text{Bottom Rectangle should be } \frac{20}{4+\pi} \text{ feet By } \frac{10}{4+\pi} \text{ feet}}$

9.



Minimize Rope = a+b

Constraints: $a=\sqrt{x^2+16}$

$b=\sqrt{(9-x)^2+16}$

$=\sqrt{x^2-18x+145}$

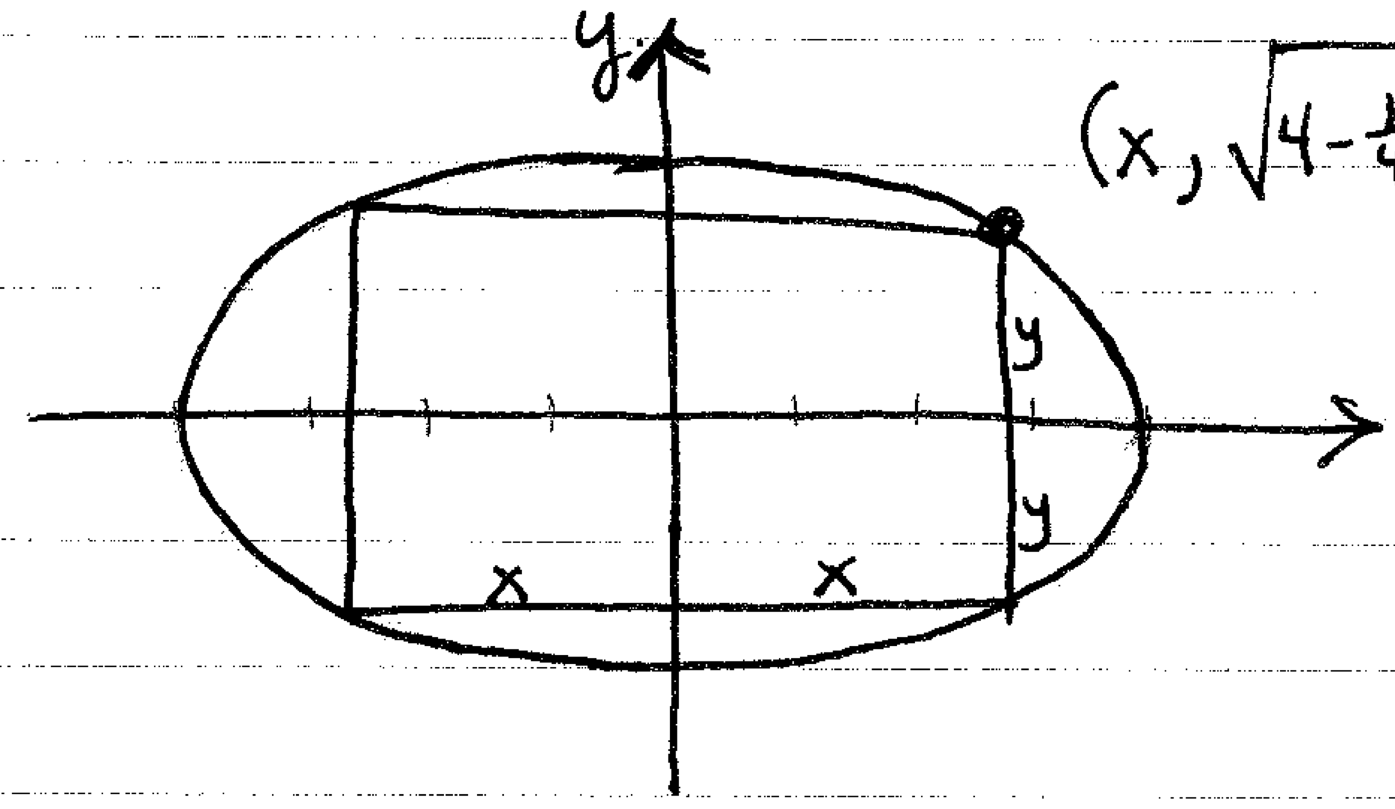
$x<9$

$$R=a+b$$
$$=\sqrt{x^2+16}+\sqrt{x^2-18x+145}$$
$$=(x^2+16)^{1/2}+(x^2-18x+145)^{1/2}$$
$$R'=\tfrac{1}{2}(x^2+16)^{-1/2}(2x)+\tfrac{1}{2}(x^2-18x+145)^{-1/2}(2x-18)$$
$$R'=\frac{x}{\sqrt{x^2+16}}+\frac{x-9}{\sqrt{x^2-18x+145}}=0$$
$$\frac{x}{\sqrt{x^2+16}}=\frac{9-x}{\sqrt{x^2-18x+145}}$$
$$x\sqrt{x^2-18x+145}=(9-x)\sqrt{x^2+16}$$

Square both sides: $x^2(x^2-18x+145)=(9-x)^2(x^2+16)$

$$x^4-18x^3+145x^2=(x^2-18x+81)(x^2+16)$$
$$x^4-18x^3+145x^2=x^4+16x^2-18x^3-288x+81x^2+1296$$
$$48x^2+288x-1296=0$$
$$48(x^2+6x-27)=0$$
$$48(x+9)(x-3)=0$$
$$\boxed{x=3\text{ feet}}$$

$\boxed{\text{Place stake 3 feet from left pole.}}$

10. 

$(x, \sqrt{4-\frac{1}{4}x^2})$

Max Area $= 4xy$

Constraint: $x^2 + 4y^2 = 16$

$y = \sqrt{4-\frac{1}{4}x^2}$

$$A = 4xy$$
$$A = 4x\left(\sqrt{4-\frac{1}{4}x^2}\right)$$
$$A = 4x\left(4-\frac{1}{4}x^2\right)^{1/2}$$

$$A' = 4x\left[\frac{1}{2}\left(4-\frac{1}{4}x^2\right)^{-1/2}\left(-\frac{1}{2}x\right)\right] + 4\sqrt{4-\frac{1}{4}x^2}$$

$$A' = \frac{-x^2}{\sqrt{4-x^2}} + 4\sqrt{4-\frac{1}{4}x^2} = 0$$

$$4\sqrt{4-\frac{1}{4}x^2} = \frac{x^2}{\sqrt{4-x^2}}$$

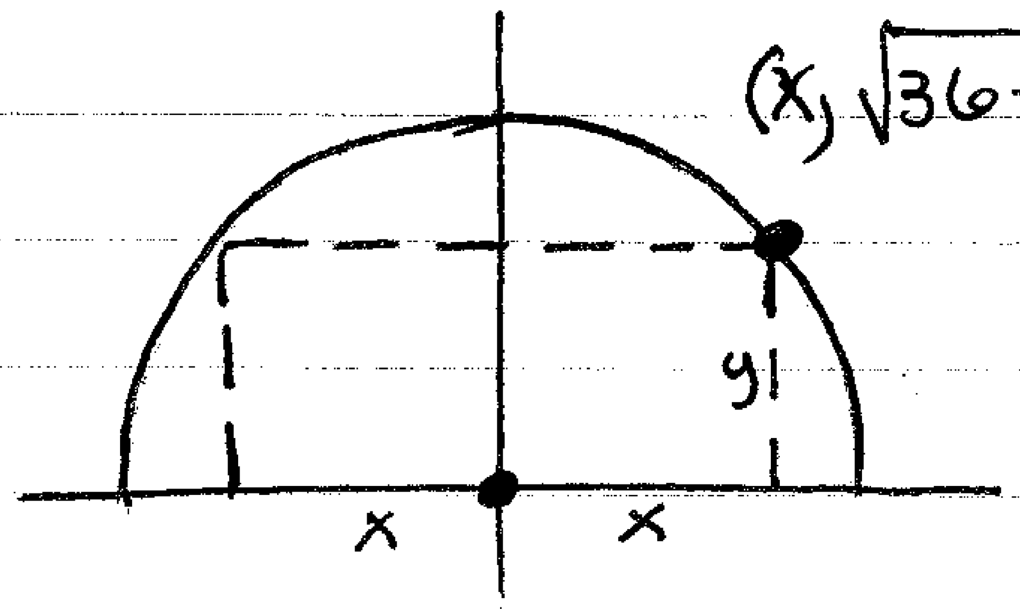
$$4\left(4-\frac{1}{4}x^2\right) = x^2$$
$$16 - x^2 = x^2$$
$$2x^2 = 16$$
$$\boxed{x = \sqrt{8} \quad y = \sqrt{2}}$$

The area of the largest rectangle is

$A = 4(\sqrt{8})\sqrt{2})$

$A = 16$ square units

11.



$(x, \sqrt{36-x^2})$

Max Area Rectangle $= 2xy$

Constraint: $x^2+y^2=36$ ← radius $= 6$!

$y = \sqrt{36-x^2}$

$$A = 2xy$$
$$A = 2x\sqrt{36-x^2}$$
$$A = 2x\,(36-x^2)^{1/2}$$

$$A' = 2x\left[\tfrac{1}{2}(36-x^2)^{-1/2}(-2x)\right] + 2\sqrt{36-x^2}$$

$$A' = \frac{x^2}{\sqrt{36-x^2}} + 2\sqrt{36-x^2} = 0$$

$$2\sqrt{36-x^2} = \frac{2x^2}{\sqrt{36-x^2}}$$

$$2x^2 = 2(36-x^2)$$
$$x^2 = 36-x^2$$
$$2x^2 = 36$$
$$x = \sqrt{18} \qquad y = \sqrt{18}$$

The area of the largest rectangle is

$A = 2xy$

$A = 2\sqrt{18}\,\sqrt{18} = 36$ square units.