Optimization #3 Name: ______________________

- *PROBLEM 1* : Build a rectangular pen with ~~three~~ four parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?

- *PROBLEM 2* : An open rectangular box with square base is to be made from 48 ft.^{2} of material. What dimensions will result in a box with the largest possible volume?

- *PROBLEM 3* : A container in the shape of a right circular cylinder with no top has surface area 5 π ft.^{2} What height h and base radius r will maximize the volume of the cylinder ?

- *PROBLEM 4* : A sheet of cardboard 4 ft. by 5 ft. will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume ?

- *PROBLEM 5* : Find the point (x, y) on the graph of $y = \sqrt{x}$ nearest the point (4, 0).

- *PROBLEM 6* : A cylindrical can is to hold 30 π m.^{3} The material for the top and bottom costs \$8/m.^{2} and material for the side costs \$5/m.^{2} Find the radius r and height h of the most economical can.

- *PROBLEM 7:* You are standing at the edge of a slow-moving river which is one mile wide and wish to return to your campground on the opposite side of the river. You can swim at 2 mph and walk at 3 mph. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is one mile from the point directly across the river from where you start your swim. What route will take the least amount of time ?

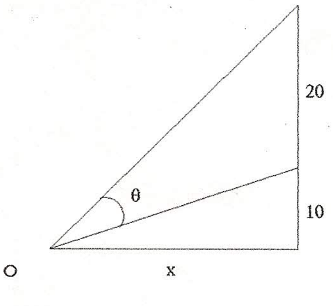
- *PROBLEM 8* : Construct a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 12 feet, what dimensions will result in the rectangle having largest possible area ?

- *PROBLEM 9* : There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

- *PROBLEM 10:* Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y=8-x^3$. (See diagram.)

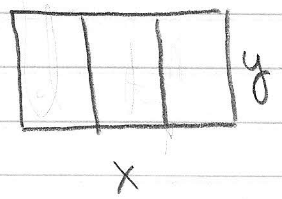
- *PROBLEM 11* : Consider a rectangle of perimeter 12 inches. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume ?

- *PROBLEM 12* : A movie screen on a wall is 20 feet high and 10 feet above the floor. At what distance x from the front of the room should you position yourself so that the viewing angle θ of the movie screen is as large as possible ? (See diagram.)



Answers to Optimization #3

1.



$P = 500$

$500 = 2x + 4y$

$2x = 500 - 4y$

$x = 250 - 2y$

$A = xy$

$A = (250 - 2y)y$

$A = 250y - 2y^2$

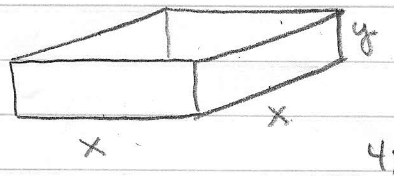
$\frac{dA}{dy} = 250 - 4y = 0$

$y = 62.5$

$x = 125$

$\boxed{\text{The dimensions of the 3 pens together are } 125' \times 62.5'.}$

2.



$SA = x^2 + 4xy$

$48 = x^2 + 4xy$

$4xy = 48 - x^2$

$y = \frac{12}{x} - \frac{x}{4}$

$V = x^2 y$

$V = x^2\left(\frac{12}{x} - \frac{x}{4}\right)$

$V = 12x - \frac{1}{4}x^3$

$\frac{dV}{dx} = 12 - \frac{3}{4}x^2 = 0$

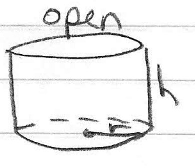
$x^2 = 16$

$x = 4$

$y = 3 - 1 = 2$

$\boxed{\text{The box should be } 4' \times 4' \times 2'.}$

3.



$SA = 5\pi$

$5\pi = \pi r^2 + 2\pi r h$

$h = (5\pi - \pi r^2)\left(\frac{1}{2\pi r}\right)$

$h = \frac{5}{2r} - \frac{1}{2}r$

$V = \pi r^2 h$

$V = \pi r^2\left(\frac{5}{2r} - \frac{1}{2}r\right)$

$V = \frac{5}{2}\pi r - \frac{1}{2}\pi r^3$

$\frac{dV}{dr} = \frac{5}{2}\pi - \frac{3}{2}\pi r^2 = 0$

$\frac{5}{2}\pi = \frac{3}{2}\pi r^2$

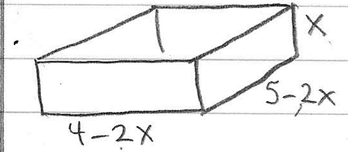
$\frac{5}{3} = r^2$

$r = \sqrt{\frac{5}{3}} \approx 1.291$

$h = 1.291$

$\boxed{\begin{array}{l}\text{The height} = 1.291 \text{ ft} \\ \text{The radius} = 1.291 \text{ ft}\end{array}}$

4.



$V = (4-2x)(5-2x)x$

$V = (20 - 18x + 4x^2)x$

$V = 4x^3 - 18x^2 + 20x$

$\frac{dV}{dx} = 12x^2 - 36x + 20 = 0$

$4(3x^2 - 9x + 5) = 0$

$x = \frac{9 \pm \sqrt{81-60}}{6} = \frac{9 \pm \sqrt{21}}{6} \approx \cancel{2.234};\ 0.736$

$\boxed{\text{The box will be about } 2.528' \times 3.528' \times 0.736'.}$

5. $(x, \sqrt{x})$

$(4, 0)$

$D = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$D = \sqrt{x^2 - 8x + 16 + x}$

$D = (x^2 - 7x + 16)^{1/2}$

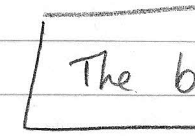
$\frac{dD}{dx} = (2x - 7)(\tfrac{1}{2})(x^2 - 7x + 16)^{-1/2} = 0$

$2x = 7$

$x = 7/2, \quad y = \sqrt{\frac{7}{2}} = \frac{\sqrt{14}}{2}$

$\boxed{\text{The point nearest } y = \sqrt{x} \text{ is } \left(\tfrac{7}{2}, \sqrt{\tfrac{7}{2}}\right).}$

6.



$V = 30\pi \text{ m}^3$

$30\pi = \pi r^2 h$

$h = \frac{30\pi}{\pi r^2}$

$h = \frac{30}{r^2}$

$Cost = 2(\pi r^2)(\$8) + 2\pi r h(\$5)$

$C = 16\pi r^2 + 10\pi r\left(\frac{30}{r^2}\right)$

$C = 16\pi r^2 + 300\pi r^{-1}$

$\frac{dC}{dr} = 32\pi r - \frac{300\pi}{r^2} = 0$

$32\pi r = \frac{300\pi}{r^2}$

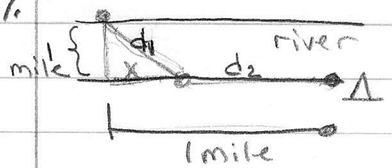
$r^3 = \frac{300\pi}{32\pi} = \frac{75}{8}$

$r = \sqrt[3]{\frac{75}{8}} \approx \boxed{2.109 \text{ m}}$

$h \approx 6.745$

The cylinder should have
radius ≈ 2.109 m
height ≈ 6.745 m.

7.



$d_1 = \sqrt{1+x^2}$ $\quad$ $d_2 = 1 - x$

$r_1 = 2mph$ $\quad$ $r_2 = 3mph$

$\sqrt{1+x^2} = 2t$ $\quad$ $1-x = 3t$

$\boxed{t_1 = \frac{1}{2}\sqrt{1+x^2}}$ $\quad$ $t_2 = \frac{1}{3} - \frac{1}{3}x$

$$\text{total Time} = \frac{1}{2}(1+x^2)^{1/2} + \frac{1}{3} - \frac{1}{3}x$$

$$\frac{dT}{dx} = 2x\left(\frac{1}{4}\right)(1+x^2)^{-1/2} - 1/3 = 0$$

$$\frac{x}{2\sqrt{1+x^2}} = \frac{1}{3}$$

$$3x = 2\sqrt{1+x^2}$$

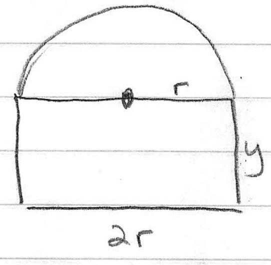
$$9x^2 = 4(1+x^2) = 4 + 4x^2$$

$$5x^2 = 4$$

$$x = \sqrt{\frac{4}{5}} \approx 0.894 \text{ miles}$$

$\boxed{\text{Swim to the point 0.894 miles from the far left, then walk.}}$

8.



$12 = 2r + 2y + \pi r$

$2y = 12 - 2r - \pi r$

$y = 6 - r - \frac{\pi}{2}r$

$A = 2ry$

$A = 2r\left(6 - r - \frac{\pi}{2}r\right)$

$A = 12r - 2r^2 - \pi r^2$

$$\frac{dA}{dr} = 12 - 4r - 2\pi r = 0$$

$$12 = r(4 + 2\pi)$$

$$r = \frac{12}{4+2\pi} = \frac{6}{2+\pi}$$

$$\approx 1.167$$

$\boxed{\text{The dimensions of the rectangle should be about } 2.334' \times 3'.}$

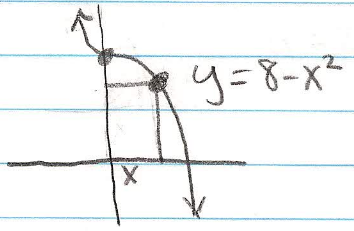
$y = 3$

9. Total output $= 50(800-10x) + x(800-10x)$ $\quad x = \#$ trees added

$$T = 40{,}000 - 500x + 800x - 10x^2$$
$$T = 40{,}000 + 300x - 10x^2$$
$$\frac{dT}{dx} = 300 - 20x = 0$$
$$x = 15 \text{ trees}$$

$\boxed{\text{15 trees should be added.}}$

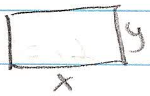
10.



$$A = xy$$
$$A = x(8-x^2)$$
$$A = 8x - x^3$$
$$\frac{dA}{dx} = 8 - 3x^2 = 0$$
$$x = \sqrt[3]{2} \approx 1.260$$

$\boxed{\text{The dimensions of the rectangle should be about 1.260 units} \times \text{6.4124 units}}$

11.



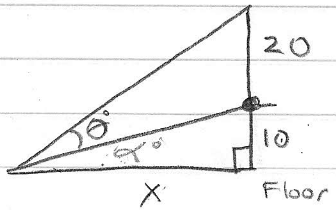
$P = 2x + 2y = 12\text{ in}$ $\quad C = x = 2\pi r$ $\quad V = \pi r^2 y$

$y = 6 - x$ $\quad r = \frac{x}{2\pi}$

$$V = \pi\left(\frac{x}{2\pi}\right)^2 y$$
$$V = \pi\left(\frac{x}{2\pi}\right)^2(6-x)$$
$$V = \left(\frac{1}{4}x^2\right)(6-x)$$
$$V = \frac{3}{2}x^2 - \frac{1}{4}x^3$$
$$\frac{dV}{dx} = 3x - \frac{3}{4}x^2 = 0$$
$$\frac{3}{4}x^2 - 3x = 0$$
$$x\left(\frac{3}{4}x - 3\right) = 0$$
$$x = 4$$
$$y = 2$$

$\boxed{\text{The rectangle should be } 4'' \times 2''.}$

12.



Maximize $\theta^\circ = \tan^{-1}\left(\frac{30}{x}\right) - \alpha^\circ$

$$\theta^\circ = \tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$$

$$\theta = \tan^{-1}(30x^{-1}) - \tan^{-1}(10x^{-1})$$

$$\frac{d\theta}{dx} = \left(\frac{1}{\frac{900}{x^2}+1}\right)\left(\frac{-30}{x^2}\right) - \left(\frac{1}{\frac{100}{x^2}+1}\right)\left(\frac{-10}{x^2}\right) = 0$$

$$\left(\frac{\cancel{x^2}}{900+x^2}\right)\left(\frac{-30}{\cancel{x^2}}\right) - \left(\frac{\cancel{x^2}}{100+x^2}\right)\left(\frac{-10}{\cancel{x^2}}\right) = 0$$

$$\frac{-30}{900+x^2} = \frac{10}{100+x^2}$$

$$3000 + 30x^2 = 9000 + 10x^2$$

$$20x^2 = 6000$$

$$x = \sqrt{300}$$

You should position yourself $\sqrt{300}$ feet from the wall.
$\theta = 30^\circ$.

$$\theta = \tan^{-1}\left(\frac{30}{\sqrt{300}}\right) - \tan^{-1}\left(\frac{10}{\sqrt{300}}\right) = \tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 60^\circ - 30^\circ = \boxed{30^\circ}$$

$$\frac{30}{\sqrt{300}} = \frac{30}{10\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$\frac{10}{\sqrt{300}} = \frac{10}{10\sqrt{3}} = \frac{1}{\sqrt{3}}$$