Free Response - Rates, Accumulation, DEQs

Calculator

1. Rate of snow accumulation $= f(t) = 7te^{\cos t}$ ft^3/hr (Volume) since midnight.

Rate of snow removal $= g(t) = \begin{cases} 0, & \text{for } 0 \le t < 6 \\ 125, & \text{for } 6 \le t < 7 \\ 108, & \text{for } 7 \le t \le 9 \end{cases}$ ft^3/hour (Volume)

Note: $f(0) = 0$

ⓐ Amt accumulated $= \int_0^6$ rate accumulated — 1: integral

$= \int_0^6 f(t)\,dt = \boxed{142.275 \text{ ft}^3}$ — 1: answer

ⓑ Snow is falling in cubic feet (Volume).
Janet is removing snow in cubic feet (volume).

Rate of change of Volume $= \begin{pmatrix}\text{Rate Accum} \\ \text{at} \\ t=8\end{pmatrix} - \begin{pmatrix}\text{Rate Removed} \\ \text{at} \\ t=8\end{pmatrix}$ — 1: answer

$= f(8) - g(8)$

$= 48.417 - 108 = \boxed{^-59.583 \text{ ft}^3/\text{hr}}$

ⓒ $h(t) =$ total amount of snow (ft^3) removed

= initial amout removed + current amount removed for any time in the interval

For $0 < t \le 6$: $h(t) = h(0) + \int_0^t g(x)\,dx = 0 + \int_0^t (0)\,dt = 0$

For $6 < t \le 7$: $h(t) = h(6) + \int_6^t g(x)\,dx = 0 + \int_6^t 125\,dx = 125t - 750$

For $7 < t \le 9$: $h(t) = h(7) + \int_7^t g(x)\,dx = 125 + \int_7^9 108\,dx = 108t - 631$

$$\boxed{h(t) = \begin{cases} 0, & \text{for } 0 \le t \le 6 \\ 125t - 750, & \text{for } 6 < t \le 7 \\ 108t - 631, & \text{for } 7 < t \le 9 \end{cases}}$$

← 1 point
← 1 point
← 1 point

ⓓ Amt Snow ON driveway at 9AM = Amt accumulated - Amt Removed

$= \int_0^9 f(t)\,dt - \int_0^9 g(t)\,dt$ ← 1: integral

$= \int_0^9 (7te^{\cos t})\,dt - \int_6^7 125\,dt - \int_7^9 108\,dt$ ← 1: $h(9)$

$= \boxed{26.335 \text{ ft}^3}$ ← 1: answer

Calculator!

2. $G(t) = 90 + 45\cos\left(\frac{t^2}{18}\right)$ = Rate gravel arrives in tons/hr, $0 \le t \le 8$.

- At $t=0$, plant has 500 tons unprocessed gravel
- $P(t)$ = amount processed gravel
- $\frac{dP}{dt}$ = rate of processing = 100 tons / hour

ⓐ $G'(5) = \frac{d}{dx}\left(90 + 45\cos\left(\frac{t^2}{18}\right)\right)\Big|_{t=5} = \boxed{-24.588 \text{ tons/hr}}$ (used calculator) 1: $G'(5)$

* At $t=5$ hours, the rate the gravel is arriving at the plant is decreasing by 24.588 tons/hour. 1: Interp with units

ⓑ Total amt. unprocessed gravel that arrives from $t=0$ to $t=8$: 1: integral

$\int_0^8 G(t)\,dt = \boxed{825.551 \text{ tons}}$ (used calculator!) 1: answer

ⓒ The rate the gravel arrives at $t=5$ hours is $G(5) = 98.141$ tons/hr which is less than the rate of processing (100 tons/hour = $\frac{dP}{dt}$).
The amount of unprocessed gravel at the plant is decreasing at $t=5$ hours. 1: Compares $G(5)$ to 100

1: conclusion

ⓓ At any time the amt. unprocessed gravel
= Initial Amt + (Amt arriving/hour − Amt processed/hour)

* $A(t) = 500 + \int_0^t (G(x) - 100)\,dx$

$A'(t) = G(x) - 100 = 0$ 1: considers $A'(t)=0$

$90 + 45\cos\left(\frac{t^2}{18}\right) - 100 = 0$ 1: answer

Use Calculator: $t = 4.923$ 1: justification

Check endpoints and local max:

→ At $t=0$: $A(t) = 500$

→ At $t=4.923$: $A(t) = 635.376$

→ At $t=8$: $A(t) = 525.551$

The maximum amount of unprocessed gravel during the hours of operation on this workday is 635.376 tons.

No Calculator!

3. W = total amt solid waste stored. $W(0) = 1400$ tons

$\frac{dW}{dt} = \frac{1}{25}(W-300)$ = Rate solid waste arriving.

t = years (for 20 years)

ⓐ Tangent line at $t=0$: $y - y_1 = m(x - x_1)$

$(x_1, y_1) = (0, 1400)$

$m = \left.\frac{dW}{dt}\right|_{x=0} = \frac{1}{25}(W(0)-300) = \frac{1}{25}(1400-300) = \frac{1100}{25} = 44$

1: $\frac{dW}{dt}$ at $t=0$

$y - 1400 = 44(x - 0)$

$\boxed{y = 44x + 1400}$

Appx at $t = \frac{1}{4}$: $y = 44(\frac{1}{4}) + 1400$

1: answer

$y = 11 + 1400 = \boxed{1411 \text{ tons}}$

ⓑ $\frac{dW}{dt} = \frac{1}{25}(W-300) = \frac{1}{25}W - 12$

$\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{25}(\frac{1}{25}W - 12) = \boxed{\frac{1}{625}W - \frac{12}{25}}$

1: $\frac{d^2W}{dt^2}$

Because $W \geq 1400$, $\frac{d^2W}{dt^2} > 0$, meaning W is concave up on the interval. The tangent line will be below W, therefore the answer in part a is an underestimate.

1: answer with reason

ⓒ $\frac{dW}{dt} = \frac{1}{25}(W-300)$

1: Sep of Var

1: antiderivs

1: "c"

$\int \frac{1}{W-300}\,dW = \int \frac{1}{25}\,dt$

$\ln|W-300| = \frac{1}{25}t + C$ → $\ln|W-300| = \frac{1}{25}t + \ln 1100$

Because $W(0) = 1400$

$W - 300 = e^{\frac{1}{25}t + \ln 1100}$

1: uses initial cond

$\ln|1400-300| = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$C = \ln 1100$

$\boxed{W = 1100e^{\frac{1}{25}t} + 300}$

1: solves for W

$0 \leq t \leq 20$

(c) Note: max 2/5 [1-1-0-0-0] if no "c"; 0/5 if no Sep. of Var.

No Calculator

4. $\frac{dB}{dt} = \frac{1}{5}(100-B) = \underline{\text{Rate}}$ bird gains weight / day $B(0)=20g$.

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(100-40) = \frac{60}{5} = 12 \text{ g/day}$ 1: uses $\frac{dB}{dt}$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(100-70) = \frac{30}{5} = 6 \text{ g/day}$ 1: answer with reason

The bird gains weight faster when it weighs 40g than when it weighs 70 g because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$.

(b) $\frac{dB}{dt} = \frac{1}{5}(100-B) = 20 - \frac{1}{5}B$

$\boxed{\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt}} = -\frac{1}{5}(20-\frac{1}{5}B) = \boxed{-4+\frac{1}{25}B}$ 1: $\frac{d^2B}{dt^2}$ in terms of B

For $20 \le B < 100$, $\frac{d^2B}{dt^2} < 0$, so the graph should be concave down. The graph shown has a part that is concave up. 1: explanation

(c) $\frac{dB}{dt} = \frac{1}{5}(100-B)$

$\left(\frac{1}{100-B}\right)dB = \frac{1}{5}dt$ 1: Sep of Var

$-\ln|100-B| = \frac{1}{5}t + C$ 1: antideriv.

$B(0) = 20$ 1: "c"

$-\ln|100-20| = C$ 1: uses initial condition

$C = -\ln 80$

therefore, 1: Solves for B

$-\ln|100-B| = \frac{1}{5}t - \ln 80$

$e^{+\ln|100-B|} = e^{-\frac{1}{5}t+\ln 80}$

$100-B = 80e^{-\frac{1}{5}t}$

$$\boxed{B = 100 - 80e^{-\frac{1}{5}t}}$$

Max: 2/5 if no "c"

0/5 if no sep. of Var

No Calculator

5. $\frac{dy}{dx} = e^{y}(3x^2 - 6x)$

(a) $y - y_1 = m(x - x_1)$ $(x_1, y_1) = (1, 0)$

$m = \left.\frac{dy}{dx}\right|_{\substack{x=1 \\ y=0}} = e^{0}(3(1)^2 - 6(1)) = -3$

$y - 0 = -3(x - 1)$

$\boxed{y = -3x + 3}$

$f(1.2) \approx -3(1.2) + 3 = -3.6 + 3 = \boxed{-0.6}$

1: $\frac{dy}{dx}$ at point $(x, y) = (1, 0)$

1: equation of tangent line

1: appx

(b) $\frac{dy}{dx} = e^{y}(3x^2 - 6x)$ point $(1, 0)$

$\frac{1}{e^y} dy = (3x^2 - 6x)dx$

$-e^{-y} = x^3 - 3x^2 + C$

at $(1, 0)$

$-1 = 1 - 3 + C$

$1 = C$

$-e^{-y} = x^3 - 3x^2 + 1$

$e^{-y} = -x^3 + 3x^2 - 1$

$\ln e^{-y} = \ln(-x^3 + 3x^2 - 1)$

$-y = \ln(-x^3 + 3x^2 - 1)$

$\boxed{y = -\ln(-x^3 + 3x^2 - 1)}$

1: Sep of Var

2: Antiderivs

1: "C"

1: uses initial cond

1: solves for y

Note:
max 3/6 if no "C"
0/6 if no separation of variables

No Calculator!

6. $f(x) = \sqrt{25-x^2}$ for $-5 \le x \le 5$

(a) $f(x) = (25-x^2)^{1/2}$ $\quad f'(x) = -2x\left(\frac{1}{2}(25-x^2)^{-1/2}\right)$ 2: $f'(x)$

(b) $f'(x) = \frac{-x}{\sqrt{25-x^2}}$ $\quad -5 \le x \le 5$

$f'(-3) = \frac{3}{\sqrt{25-9}} = \frac{3}{4}$ 1: $f'(-3)$

$f(-3) = \sqrt{25-9} = 4$ 1: answer

$$\boxed{y-4 = \frac{3}{4}(x+3)}$$

(c) $g(x) = \begin{cases} f(x), & \text{for } -5 \le x \le -3 \\ x+7, & \text{for } -3 < x \le 5 \end{cases}$ 1: considers one-sided limits

1: answer with explanation

$\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} f(x) = 4$

$\lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} (x+7) = 4$

$g(-3) = f(-3) = 4$

Because, $\lim_{x \to -3} g(x) = g(-3)$, $g(x)$ is continuous at $x = -3$.

(d) $\int_0^5 x(25-x^2)^{1/2}\,dx$ $\quad u = 25-x^2$

$du = -2x\,dx$ 2: antiderivative

1: answer

$-\frac{1}{2}\int_0^5 (25-x^2)^{1/2}(-2x\,dx) = -\frac{1}{2}\int_0^5 u^{1/2}\,du$

$= \left(-\frac{1}{2}\right) u^{3/2}\left(\frac{2}{3}\right)\Big|_0^5 = -\frac{1}{3}(25-x^2)^{3/2}\Big|_0^5$

$= -\frac{1}{3}\left(-25^{3/2}\right) = \boxed{\frac{125}{3}}$

Calculator

7. $\boxed{A(t) = 6.687(0.931)^t}$ pounds: Amt grass clipping remaining in bin

ⓐ $\dfrac{A(30)-A(0)}{30-0} = \dfrac{0.783-6.687}{30} = \boxed{-0.197 \text{ lbs/day}}$ — 1: answer with units

ⓑ $A'(15) = \boxed{-0.164}$ — 1: $A'(15)$

The amount of grass clippings in the bin is decreasing at a rate of 0.164 lbs/day at time $t=15$. — 1: interpretation

ⓒ $A(t) = 6.687(0.931)^t = \dfrac{1}{30-0}\displaystyle\int_0^{30} \left(6.687(0.931)^t\right)dt$ — 1: $\frac{1}{30}\int_0^{30} A(t)dt$

$6.687(0.931)^t = 2.753$

$6.687(0.931)^t - 2.753 = 0$

graph, find zero: $\boxed{t = 12.402}$ days — 1: answer

ⓓ $L(t)$ = linear appx to A at $t=30$

= tangent line approximation

$y - y_1 = m(t - t_1)$

$m = A'(t) \qquad (t_1, y_1) = (30, A(30))$

$m = A'(30)$

$y - A(30) = A'(30)(t-30)$

$y = A(30) + A'(30)(t-30)$: Linear Appx to A at $t=30$

OR

$L(t) = A(30) + A'(30)(t-30)$ — 2: expression for $L(t)$

$\frac{1}{2} = A(30) + A'(30)(t-30)$

$0 = -\frac{1}{2} + 0.783 + (-0.056)(t-30)$

graph, find zero: $\boxed{t = 35.054 \text{ days}}$ — 1: $L(t) = \frac{1}{2}$

1: answer