Free Response - Particle Motion

Calculator

1. $v(t) = 2\sin\left(e^{t/4}\right) + 1$

$a(t) = \frac{1}{2} e^{t/4} \cos\left(e^{t/4}\right)$

$x(0) = 2$

(a) Speed of particle increases when $v(t)$ and $a(t)$ have the same sign and decreases when $v(t)$ and $a(t)$ have opposite signs.

$v(5.5) = -0.453$

$a(5.5) = -1.359$

$v(t)$ and $a(t)$ have the same sign at $t = 5.5$ therefore speed is increasing.

2: conclusion with reason

(b) Average velocity. You've been given the velocity function so

$$\text{Avg } v(t) = \frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{6-0}\int_0^6 v(t)\,dt = \boxed{1.949}$$

1: integral
1: answer

(c) $$\text{Total Distance} = \int_a^b |v(t)|\,dt = \int_0^6 |v(t)|\,dt = \boxed{12.573}$$

1: integral
1: answer

(d) Particle changes direction when $v(t)$ crosses the t-axis. (when $v(t)$ changes signs).

(Graph $v(t)$ - find root)

$v(t)$ crosses t-axis at $t = 5.196$.

The position of the particle:

$$x(0) + \int_0^{5.196} v(t)\,dt = 2 + 12.135 = \boxed{14.135}$$

1: considers $v(t) = 0$
1: integral
1: answer

No calculator

2. $v(t) = \cos\left(\frac{\pi}{6}t\right) \qquad s(0) = -2$

ⓐ The particle moves left when $v(t) < 0$.

$v(t) = \cos\left(\frac{\pi}{6}t\right) = 0$ — 1: considers $v(t) = 0$

when $\frac{\pi}{6}t = \frac{\pi}{2}$ OR $\frac{\pi}{6}t = \frac{3\pi}{2}$

$t = 3 \qquad t = 9$ — 1: interval

$v(t)$: $+$ on $(0, 3)$, $-$ on $(3, 9)$, $+$ after 9

0 3 9

$\boxed{\text{The particle moves left } 3 < t < 9.}$

ⓑ Total Distance $= \int_a^b |v(t)|\,dt = \boxed{\int_0^6 \left|\cos\left(\frac{\pi}{6}t\right)\right| dt}$ — 1: answer

ⓒ $\boxed{a(t) = v'(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)}$ — 1: $a(t)$

Speed increases when $a(t)$ and $v(t)$ have the same sign. — 2: conclusion with reason

At $t = 4$: $v(4) = \cos\frac{2\pi}{3} = -\frac{1}{2}$

$a(4) = -\frac{\pi}{6}\sin\frac{2\pi}{3} = \left(-\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right)$

$v(4)$ and $a(4)$ have the same sign, so the speed is increasing at $t = 4$.

ⓓ Position at $t = 4$ $= s(4) = s(0) + \int_0^4 v(t)\,dt$ — 1: antideriv

$= -2 + \int_0^4 \cos\frac{\pi}{6}t\,dt$ — 1: uses initial condition

$u = \frac{\pi}{6}t$

$du = \frac{\pi}{6}dt$

$= -2 + \frac{6}{\pi}\int_0^4 \left(\cos\frac{\pi}{6}t\right)\left(\frac{\pi}{6}dt\right)$ — 1: answer

$= -2 + \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\Big|_0^4$

$= -2 + \frac{6}{\pi}\left(\sin\frac{2\pi}{3} - \sin 0\right)$

$= -2 + \frac{6}{\pi}\left(\frac{\sqrt{3}}{2}\right)$

$= \boxed{-2 + \frac{3\sqrt{3}}{\pi}}$

No calculator

3. $v(0)=0$ $v(3)=0$ $v(5)=0$

$a(1)=0$ $a(4)=0$

$s(0)=-2$

(a) Particle is farthest to the left at the absolute minimum of $v(t)$. $v(t)<0$ for $0<t<3$ and $5<t<6$. 1: Id's $t=3$ as a candidate

At $t=0$, particle is at -2. 1: considers $\int_0^6 v(t)\,dt$

At $t=3$, $s(3)=s(0)+\int_0^3 v(t)\,dt=-2+(-8)=-10$

At $t=6$, $s(6)=s(0)+\int_0^6 v(t)\,dt=-2+(-8+3-2)=-9$

The particle is farthest left at $t=3$ when the position is $s(3)=-10$. 1: conclusion

(b) Because $v(t)$ is differentiable, $v(t)$ must be continuous.

We know $s(0)=-2$ and $s(3)=-10$. By the IVT $s(t)$ must equal -8 at some time t on $[0,3]$. 1: positions at $t=3,5,6$

We know $s(3)=-10$ and $s(5)=-2+(-8+3)=-7$. So, again, by IVT, $s(t)$ must equal -8 at some time t on $[3,5]$. 1: description of motion OR IVT

1: conclusion

We know $s(5)=-7$ and $s(6)=-9$, so, again, by IVT $s(t)$ must equal -8 at some time t on $[5,6]$.

Therefore $s(t)=-8$ for three values of t.

(c) On the interval $(2,3)$, $v(t)<0$. $a(t)$ is the slope of $v(t)$. On the interval $(2,3)$ $a(t)>0$. 1: answer with reason

Because $v(t)$ and $a(t)$ have different signs, the speed of the particle is decreasing.

(d) $a(t)$ is negative when $v(t)$ is decreasing. 1: answer

This occurs on the intervals $(0,1)$ and $(4,6)$. 1: justification

No calculator

4.

ⓐ $\int_0^{24} v(t)\,dt = 40 + 12(20) + 80$

$= 360$

The car travels 360 meters in these 24 seconds.

1: value

1: meaning with units

ⓑ $v'(4)$ = does not exist, because the left-hand derivative does not equal the right-hand derivative:

$$\lim_{t \to 4^-} \left(\frac{v(t) - v(4)}{t-4} \right) = 5 \neq \lim_{t \to 4^+} \left(\frac{v(t) - v(4)}{t-4} \right) = 0$$

1: $v'(4)$ DNE with reason

$$v'(20) = \frac{v(24) - v(16)}{24 - 16} = \frac{0 - 20}{8} = \frac{-20}{8} = \boxed{-\frac{5}{2}\ m/sec^2}$$

1: $v'(20)$

1: units

ⓒ $$a(t) = \begin{cases} 5, & 0 < t < 4 \\ 0, & 4 < t < 16 \\ -\frac{5}{2}, & 16 < t < 24 \end{cases}$$

1: find values $5, 0, -\frac{5}{2}$

Note: $a(t)$ does not exist at $t = 4$ and $t = 16$

1: Id's constants with correct intervals

ⓓ Average rate of change on $[8, 20]$: $\frac{v(20) - v(8)}{20 - 8} = \boxed{-\frac{5}{6}\ m/sec^2}$

The mean value theorem states that the function must be continuous on the closed interval and differentiable on the open interval. $v(t)$ is not differentiable at $t = 16$, therefore the mean value theorem does not apply to $v(t)$ on $[8, 20]$.

1: average rate of change of v on $[8,20]$

1: answer with explanation

calculator

5\. (a) acceleration at $t=7.5$ is slope of graph of $v(t)$ at $t=7.5$ — 1: answer

$a(7.5) = v'(7.5) = \frac{v(8)-v(7)}{8-7} = \boxed{-0.1 \text{ mi/min}^2}$ — 1: units

(b) $\int_0^{12} |v(t)|\,dt$ is the total distance, in miles, that Caren rode from $t=0$ minutes to $t=12$ minutes. — 1: meaning of integral

$\int_0^{12} |v(t)|\,dt = \int_0^{2} v(t)\,dt - \int_2^{4} v(t)\,dt + \int_4^{12} v(t)\,dt$ — 1: value of integral

$= 0.2-(-0.2)+1.4 = \boxed{1.8 \text{ miles}}$

(c) Caren turns around to go back home at time = 2 minutes. This is the time her velocity changes from positive to negative. — 1: answer, 1: reason

(d) Larry: $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$ mi/min on $[0,12]$.

Find distance

Larry $\int_0^{12} \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)dt = 1.6$ miles from school — 2: Larry's distance from school where 1: integ. 1: answer

Caren $\int_0^{12} v(t)\,dt = \int_0^{2} v(t)\,dt + \int_2^{4} v(t)\,dt + \int_4^{12} v(t)\,dt$

$= 0.2 - 0.2 + 1.4$

$= 1.4$ miles from school — 1: Caren's distance from school and conclusion

Caren lives closer to school.

No calculator

6.

(a) Average $a(t)$ over $2 \le t \le 8$: $\dfrac{f(b)-f(a)}{b-a} = \dfrac{-120-100}{8-2} = \dfrac{-220}{6} = \boxed{\dfrac{-110}{3}\ m/min^2}$

1: average acceleration

(b) $v_A(5) = 40$ and $v_A(8) = -120$, $v_A(8) < -100 < v_A(5)$

Because $v_A(t)$ is differentiable, it is continuous.

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

1: $v_A(8) < -100 < v_A(5)$

1: conclusion, using IVT

(c) $s_A(2) = 300\text{m}$

$s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = \boxed{300 + \int_2^{12} v_A(t)\,dt}$

1: position expression

$\int_2^{12} v_A(t)\,dt \approx \frac{3}{2}(100+40) + \frac{3}{2}(40-120) + \frac{4}{2}(-120-150)$

1: Trap. Sum

$= 210 - 120 - 540 = \boxed{-450}$

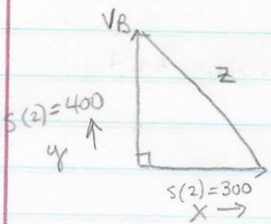
1: Position at $t=12$

Therefore:

$s_A(12) = 300 - 450 = \boxed{-150}$

At $t = 12$ minutes, Train A is approximately 150 meters west of Origin Station.

(d) $v_B(t) = -5t^2 + 60t + 25$ $\quad s(2) = 400\text{m}$ North



Find $\frac{dz}{dt}$:

$z^2 = x^2 + y^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$

2: Implicit differentiation of distance relationship

1: answer

$z = \sqrt{300^2 + 400^2} = 500$

(Note: Pythagorean Triple)

At $t=2$:

$x = 300$, $\frac{dx}{dt} = 100$

$y = 400$, $\frac{dy}{dt} = v_B(2) = -20 + 120 + 25 = 125$

$\frac{dz}{dt} = \frac{1}{500}\left(300(100) + 400(125)\right)$

$= \frac{1}{500}(30{,}000 + 50{,}000)$

$= \frac{1}{500}(80{,}000) = \boxed{160\ m/min}$

No calculator

1. ⓐ At $t=5$ $\quad a(5) \approx \frac{v(10)-v(0)}{10-0} = \frac{2.3-2.0}{10} = \frac{.3}{10} = \boxed{.03 \ m/sec^2}$ 1: answer

ⓑ $\int_0^{60} |v(t)|\,dt$ is the total distance, in meters, that Ben rides from $t=0$ sec to $t=60$ sec. 1: meaning of integral

LRAM:

$$\int_0^{60} |v(t)|\,dt \approx 10(2) + 30(2.3) + 20(2.5)$$ 1: appx.

$$= 20 + 69 + 50 = \boxed{139 \text{ meters}}$$

ⓒ $v(40) = 2.5 \qquad v(60) = 4.6$

MVT: $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{49-9}{60-40} = \frac{40}{20} = 2$ 1: difference quotient

The Mean Value theorem implies there is a time t in the interval $(40, 60)$ such that $v(t) = 2$. 1: conclusion w/justification

ⓓ $(L(t))^2 = 12^2 + (B(t))^2$
(Pythagorean Theorem)

Derivative: $2(L(t))(L'(t)) = 2(B(t))(B'(t))$ 1: derivatives

Find $L'(t)$: 1: uses $B'(t) = v(t)$

$$L'(t) = \cancel{2}(B(t))(B'(t))\left(\frac{1}{\cancel{2}L(t)}\right)$$ 1: answer

$$L'(40) = B(40)\,B'(40)\left(\frac{1}{L(40)}\right)$$

$L(40) = \sqrt{12^2 + (B(40))^2}$
$= \sqrt{144 + 81}$
$= \sqrt{225}$
$= 15$

$$= (9)(2.5)\left(\frac{1}{15}\right)$$ 1: units in ⓐ or ⓑ correct

$$= \frac{7.5}{5}$$

$$= \left(\frac{15}{2}\right)\left(\frac{1}{5}\right)$$

$$= \boxed{\frac{3}{2} \text{ m/sec}}$$