Free Response - Function Analysis Solution

#1 No Calculator

ⓐ f has a local minimum when the graph of f' changes from negative to positive. This occurs at $x=6$.

1: answer with justif.

ⓑ The absolute minimum of f occurs at either an endpoint or at the local minimum $x=6$.

We know $f(8)=4$:

$f(0) = f(8) - \int_0^8 f'(x)\,dx = 4 - 12 = -8$

$f(6) = f(8) - \int_6^8 f'(x)\,dx = 4 - 7 = -3$

$f(8) = 4$

The absolute minimum value of f is -8 and occurs at $x=0$.

1: considers $f(0), f(6), f(8)$
1: answer
1: justification

ⓒ The graph of f is concave down and increasing when f' is decreasing and positive. This occurs on the intervals $(0,1)$ and $(3,4)$.

1: answer
1: explanation

ⓓ $g(x) = (f(x))^3$

$f(3) = -\frac{5}{2}$ Find slope of tangent of g at $x=3$. (Find $g'(3)$).

2: $g'(x)$
1: answer

$g'(x) = (f'(x))(3(f(x))^2)$

$g'(3) = (f'(3))(3(f(3))^2)$

$= (4)(3(-\frac{5}{2})^2)$

$= 12(\frac{25}{4})$

$= \boxed{75}$

$f'(3)$ on graph $= 4$

$f(3) = -5/2$ (given)

No calculator

#2

$g(x) = \int_1^x f(t)\,dt$

$m = \frac{-1}{2}$

$y - 0 = -\frac{1}{2}(x-1) \rightarrow y = -\frac{1}{2}x + \frac{1}{2}$

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt = -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \boxed{\frac{\pi}{2} - \frac{3}{2}}$

1: $g(2)$
1: $g(-2)$

(b) $g'(-3) = f(-3) = \boxed{2}$ $\left(g'(x) = f(x)!\right)$

$g''(-3) =$ slope of $f(x)$ at $x = -3$: $\frac{3-1}{-2+4}$

$g''(-3) = \boxed{1}$

1: $g'(-3)$
1: $g''(-3)$

(c) The graph of g has a horizontal tangent when $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

At $x = -1$, $g'(x) = f(x)$ changes from positive to negative, therefore g has a relative maximum at $x = -1$.

At $x = 1$, $g'(x) = f(x)$ does not change signs, so g has neither a relative minimum or a relative maximum there.

1: considers $g'(x) = 0$
1: $x = -1, 1$
1: answers w/ justif.

(d) The graph of g has a point of inflection when the graph of f changes from increasing to decreasing or decreasing to increasing. This occurs at $x = --2$, $x = 0$, and $x = 1$.

1: answer
1: explanation

#3

No Calculator

$$g(x) = \int_{-3}^{x} f(t)\,dt$$

ⓐ $g(3) = \int_{-3}^{3} f(t)\,dt = 6+4-1 = \boxed{9}$

1: answer

ⓑ The graph of g is increasing when $f(t) > 0$ and concave down when $f(t)$ is decreasing. $f(t) > 0$ and decreasing on the intervals $(-5,-3)$ and $(0,2)$.

1: answer
1: reason

ⓒ $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$$h'(x) = \frac{5x g'(x) - 5g(x)}{25x^2} \qquad h'(3) = \frac{5(3)\,g'(3) - 5g(3)}{25(3)^2}$$

The graph is of $g'(x)$, so $g'(3) = -2$.
$g(3)$ was found in ⓐ.

$$= \frac{15(-2) - 5(9)}{25(9)}$$

$$= \frac{-30-45}{225}$$

$$= \frac{-75}{225} = \boxed{-\frac{1}{3}}$$

2: $h'(x)$
1: answer

ⓓ $p(x) = f(x^2-x)$ Find slope of line tangent to graph of p where $x=-1$.
(Find $p'(-1)$)

$p'(x) = (2x-1)\,f'(x^2-x)$ (chain rule)
$p'(-1) = (-3)(f'(2))$
$= -3(-2)$
$= \boxed{6}$

$\left\{ \begin{array}{l} f'(2) = \text{slope of } f(x) \text{ at } x=2 \\ f'(2) = -2 \end{array} \right.$

2: $p'(x)$
1: answer

#4 No calculator

$$g(x) = 2x + \int_0^x f(t)\,dt \qquad f \text{ is continuous}$$

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = \boxed{-6 - \frac{9}{4}\pi}$ — 1: $g(-3)$

$g'(x) = \boxed{2 + f(x)}$ — 1: $g'(x)$

$g'(-3) = 2 + f(-3) = 2 + 0 = \boxed{2}$ — 1: $g'(-3)$

(b) g has an absolute maximum at an endpoint or at a relative maximum on $[-4,3]$.

g has a relative maximum when $g'(x)=0$:

$g'(x) = 2 + f(x) = 0$ when $f(x) = -2$. — 1: considers $g'(x)=0$

$f(x) = -2$ at $x = \frac{5}{2}$. — 1: identifies $x = 5/2$

$g(-4) = -8 + \int_0^{-4} f(t)\,dt = -8 + \left(-\frac{9}{4}\pi + \frac{1}{4}\pi\right) = -8 - 2\pi$

$g\left(\frac{5}{2}\right) = 5 + \int_0^{5/2} f(t)\,dt = 5 + \frac{9}{4} - 1 = 6\frac{1}{4}$

$g(3) = 6 + \int_0^{3} f(t)\,dt = 6 + \frac{9}{4} - \frac{9}{4} = 6$ — 1: answer w/ justif.

Therefore g has an absolute maximum at $x = \frac{5}{2}$.

(c) $g(x)$ has a point of inflection when f' changes signs. This occurs at $x = 0$. — 1: answer w/ reason

(d) Average Rate of change: $\frac{f(3) - f(-4)}{3+4} = \boxed{-\frac{2}{7}}$ — 1: avg rate of change

The MVT states that f must be differentiable on the interval $(-4,3)$. f is not differentiable at $x = -3$ or at $x = 0$. — 1: explanation