Continuity + Tangents from Tuesday 4/24

5. a. A relative minimum occurs when $f'(x)=0$ and the sign of $f'$ changes from negative to positive. This occurs at $x=1$.

b. Average rate of change between $x=-1$ and $x=0$:

$$\frac{0-0}{1+1}=0$$

$f$ is twice differentiable, therefore $f'$ is continuous and differentiable. By the Mean Value Theorem, there is a value $c$ such that $f''(c)=0$.

c. $h'(x)=\frac{1}{f(x)}\left(f'(x)\right)$

$$h'(3)=\frac{1}{f(3)}\left(f'(3)\right)=\left(\tfrac{1}{7}\right)\left(\tfrac{1}{2}\right)=\boxed{\tfrac{1}{14}}$$

d. $$\int_{-2}^{3} f'(g(x))\,g'(x)\,dx = f(g(x))\Big|_{-2}^{3} = f(g(3))-f(g(-2))$$

$$=f(1)-f(-1)$$

$$=\boxed{-6}$$

6. a. point $(-1,1)$ slope $= \frac{dy}{dx} = \left.\frac{y}{3y^2-x}\right|_{(-1,1)} = \frac{1}{4}$

$$\boxed{y-1 = \frac{1}{4}(x+1)}$$

b. $\frac{dy}{dx} = \frac{y}{3y^2-x}$ $\frac{dy}{dx}$ has a vertical slope when $3y^2-x=0$

or

$x = 3y^2$

Since $y^3 - xy = 2$

$y^3-(3y^2)y = 2$

$-2y^3 = 2$

$y = -1$

and then $(-1)^3 - x(1) = 2$

$x = 3$

$\boxed{\text{The tangent line to the curve is vertical at the point } (3,-1).}$

c. $\frac{d^2y}{dx^2} = \frac{(3y^2-x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx}-1\right)}{(3y^2-x^2}$

$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1}{4}$ therefore

$\frac{d^2y}{dx^2} = \frac{(3+1)\left(\frac{1}{4}\right) - 6\left(\frac{1}{4}\right)+1}{16} = \boxed{\frac{1}{32}}$

6. a. $k(x) = f(g(x))$

$k(3) = f(g(3)) = f(6) = 4$ point $(3, 4)$

$k'(x) = f'(g(x))\, g'(x)$

$k'(3) = f'(g(3))\, g'(3) = f'(6)\, g'(3) = 10$ = slope

$$\boxed{y - 4 = 10(x-3)}$$

b. $h'(x) = \dfrac{f(x)\, g'(x) - g(x)\, f'(x)}{(f(x))^2}$

$h'(1) = \dfrac{(-6)(8) - (2)(3)}{36} = \boxed{\dfrac{-3}{2}}$

c. $\int_1^3 f''(2x)\,dx = \frac{1}{2}\int_1^3 2f''(2x)\,dx$

$= \frac{1}{2} f'(2x)\Big|_1^3 = \frac{1}{2}\left(f'(6) - f'(1)\right) = \boxed{\frac{7}{2}}$

4. a. point $(0, 91°C)$

slope $= \frac{dh}{dt} = -\frac{1}{4}(91-27) = -16$

$\boxed{H - 91 = -16t}$

$H(t) = -16t + 91$

$\boxed{H(3) \approx 43°C}$

b. $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = -\frac{1}{4}\left(-\frac{1}{4}(H-27)\right)$

when $t = 3$, $H(3) = 43°C$ (from part a)

$\frac{d^2H}{dt^2}$ at $t=3$ : $-\frac{1}{4}\left(-\frac{1}{4}(43-27)\right) > 0$ so $H$ is concave up.

$H$ is concave up at $t=3$, therefore the answer in part (a) is an underestimate.

c. $\frac{dG}{dt} = -(G-27)^{2/3}$

$(G-27)^{-\frac{2}{3}} dG = -dt$

$3(G-27)^{1/3} = -t + C$

using $(0, 91)$: $3(91-27)^{\frac{1}{3}} = C$

$C = 12$

$3(G-27)^{1/3} = -t + 12$

$(G-27)^{1/3} = -\frac{1}{3}t + 4$

$G - 27 = \left(-\frac{1}{3}t + 4\right)^3$

$\boxed{G(t) = \left(-\frac{1}{3}t + 4\right)^3 + 27 \quad \text{for } t < 10 \text{ min}}$

$G(3) = \left(-\frac{1}{3}(3) + 4\right)^3 + 27 = \boxed{54°C}$