Calculus AB Bible

(2nd most important book in the world)
(Written and compiled by Doug Graham)

LIMITS

** When evaluating limits, we are checking around the point that we are approaching, NOT at the point.

**Every time we find a limit, we need to check from the left and the right hand side (Only if there is a BREAK at that point).

**Breaking Points are points on the graph that are undefined or where the graph is split into pieces.

Breaking Points:

1) Asymptotes (when the denominator equals 0)
2) Radicals (when the radical equals 0)
3) Holes (when the numerator and denominator equals 0)
4) Piece-wise functions (the # where the graph is split)

$\lim_{x \to a^+} f(x) =$ right hand limit $\qquad \lim_{x \to a^-} f(x) =$ left hand limit

**If left and right hand limits DISAGREE, then the limit Does Not Exist (DNE) at that point.

**If left and right hand limits AGREE, then the limit exists at that point as that value.

**Even if you can plug in the value, the limit might not exist at that point. It might not exist from the left or right side or the two sides will not agree.

For example: $f(x) = \begin{cases} 3 \text{ for } x \geq 1 \\ 1 \text{ for } x < 1 \end{cases}$ $\quad \lim_{x \to 1} f(x) = DNE$ because $\lim_{x \to 1^+} f(x) = 3$ and $\lim_{x \to 1^-} f(x) = 1$

Note: In general when doing limits, $\quad \dfrac{\#}{x \to 0} = \infty \qquad \dfrac{-\#}{x \to 0} = -\infty \qquad \dfrac{\#}{x \to \infty} = 0$

LIMITS

LIMITS AT NON-BREAKING POINTS (Very easy. Just plug in the #)

EX#1: $\lim_{x \to 1} x^3 + x - 5 = -3$ $\qquad$ **EX#2:** $\lim_{x \to 2} \sqrt{x+7} = \sqrt{9} = 3$ $\qquad$ **EX#3:** $\lim_{x \to 1} \dfrac{2x-1}{x+1} = \dfrac{1}{2}$

HOLES IN THE GRAPH $\left(\frac{0}{0}\right)$ (Factor and cancel or multiply by the conjugate and cancel, then plug in #)

EX#1: $\lim_{x \to 2} \dfrac{x^2+3x-10}{x-2} = \lim_{x \to 2} \dfrac{(x+5)(x-2)}{x-2} = \lim_{x \to 2} (x+5) = 7$

EX#2: $\lim_{x \to -2} \dfrac{\sqrt{x+11}-3}{x+2} = \lim_{x \to -2} \dfrac{\left(\sqrt{x+11}-3\right)\left(\sqrt{x+11}+3\right)}{(x+2)\left(\sqrt{x+11}+3\right)} = \lim_{x \to -2} \dfrac{\overset{1}{\cancel{x+2}}}{\underset{1}{\cancel{(x+2)}}\left(\sqrt{x+11}+3\right)} = \dfrac{1}{6}$

RADICALS (You must first check that the limit exists on the side(s) you are checking)

If a # makes a radical negative, the limit will not exist at that #.

When we check at the breaking point (the # that makes the radical zero) there are two possible answers:

1) 0 if the limit works from the side that you are checking.

2) DNE if the limit does not work from the side that you are checking.

EX #1: $\lim_{x\to 3^-} \sqrt{3-x} =$ Since the limit exists from the left at 3 we can plug in 3, then $\lim_{x\to 3^-} \sqrt{3-x} = 0$

EX #2: $\lim_{x\to 5^+} \sqrt{5-x} =$ Since the limit does not exist from the right at 5, then $\lim_{x\to 5^+} \sqrt{5-x} =$ DNE

EX #3: $\lim_{x\to -2} \sqrt{x+2} = DNE$ because $\lim_{x\to -2^+} \sqrt{x+2} = 0$ $\lim_{x\to -2^-} \sqrt{x+2} = DNE$ (both sides don't agree).

ASYMPTOTES $\left(\#/_0\right)$ (Since the point DNE we have to check a point that is close on the side we are approaching)

There are three possible answers when checking at the breaking point (the # that makes bottom = zero)

1) ∞ $\to$ If we get a positive answer the limit approaches ∞

2) $-\infty$ $\to$ If we get a negative answer the limit approaches $-\infty$

3) DNE $\to$ If we get a positive answer on one side and a negative answer on the other side, then the limit DNE

EX #1: $\lim_{x\to 5^-} \frac{3}{5-x} =$ Check 4.9 which gives a positive answer so $\lim_{x\to 5^-} \frac{3}{5-x} = \infty$

EX #2: $\lim_{x\to -8} \frac{7}{x+8} =$ Check -8.1 which gives a negative answer and -7.9 which gives a positive answer

so $\lim_{x\to -8} \frac{7}{x+8} = DNE$ because the two sides do not agree.

EX #3: $\lim_{x\to -3^+} \frac{3x}{x+3} =$ Check -2.9 which gives a negative answer so $\lim_{x\to -3^+} \frac{3x}{x+3} = -\infty$

EX #4: $\lim_{x\to 9} \frac{10}{(x-9)^2} = \infty$ because $\lim_{x\to 9^+} \frac{10}{(x-9)^2} = \infty$ $\lim_{x\to 9^-} \frac{10}{(x-9)^2} = \infty$ (both sides agree).

TRIG. FUNCTIONS

FACTS: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$ $\lim_{x\to 0} \frac{1-\cos ax}{bx} = 0$ $\lim_{x\to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

EX#1: $\lim_{x\to 0} \frac{\sin x \tan x}{x^2} = \lim_{x\to 0} \frac{\sin x \tan x}{x \cdot x} = 1 \cdot 1 = 1$

EX#2: $\lim_{x\to 0} \frac{3\sin 3x}{8x} = 3 \cdot \frac{3}{8} = \frac{9}{8}$

EX#3: $\lim_{x\to 0} \frac{6\sin x \cos x}{5x} = \lim_{x\to 0} \frac{6}{5} \frac{\sin x}{x} \frac{\cos x}{1} = \frac{6}{5} \cdot 1 \cdot 1 = \frac{6}{5}$

EX#4: $\lim_{x\to \pi/2} \frac{5\tan 3x}{x} = 5 \cdot \frac{3}{1} = 15$

PIECE - WISE FUNCTIONS

$$f(x)=\begin{cases} 3-x & x<-3 \\ 2x+1 & -3\le x<4 \\ 9 & x\ge 4 \end{cases}$$

The breaking points are -3 and 4.

EX #1: $\lim\limits_{x\to -3^+} f(x) = -5$ $(Check\ in\ x > -3)$

EX #2: $\lim\limits_{x\to 4^+} f(x) = 9$ $(Check\ in\ x > 4)$

EX #3: $\lim\limits_{x\to -3^-} f(x) = 6$ $(Check\ in\ x < -3)$

EX #4: $\lim\limits_{x\to 4^-} f(x) = 9$ $(Check\ in\ x < 4)$

EX #5: $\lim\limits_{x\to -3} f(x) = DNE$ (Both sides don't agree)

EX #6: $\lim\limits_{x\to 4} f(x) = \ 9$ (Both sides agree)

These next three limits are not at breaking points, so we just plug in the numbers.

EX #7: $\lim\limits_{x\to 7^+} f(x) = 9$

EX #8: $\lim\limits_{x\to -5} f(x) = \ 3 - {}^{-}5 = 8$

EX #9: $\lim\limits_{x\to 2} f(x) = \ 2(2)+1 = 5$

LIMITS THAT APPROACH INFINITY

Check the powers of the numerator and denominator.

1) If the denominator (bottom) is a bigger power the limit = 0.

2) If the numerator (top) is a bigger power the limit = ∞ or $-\infty$.

3) If powers are the same the limit = $\dfrac{\text{coefficient of the highest power of numerator}}{\text{coefficient of the highest power of denominator}}$

EX#1: $\lim\limits_{x\to\infty} \dfrac{3-5x^2}{13x^2+1} = \dfrac{-5}{13}$

EX#2: $\lim\limits_{x\to\infty} \dfrac{9-x^3}{x^2} = -\infty$

EX#3: $\lim\limits_{x\to\infty} \dfrac{1}{6-x} = \ 0$

EX#4: $\lim\limits_{x\to\infty} \dfrac{7-x}{x-7} = -1$

EX#5: $\lim\limits_{x\to-\infty} \dfrac{2x-5}{9x+1} = \ \dfrac{2}{9}$

EX#6: $\lim\limits_{x\to-\infty} \dfrac{2x^5+3}{7x^2-5} = -\infty$

EX#7: $\lim\limits_{x\to-\infty} \dfrac{5-x}{3x^2+1} = \ 0$

EX#8: $\lim\limits_{x\to\infty} 3 = \ 3$

FINDING VERTICAL ASYMPTOTES AND HOLES

A **vertical asymptote** is the # that makes only the denominator = 0.

A **hole** occurs at the points that make the numerator and denominator = 0 at the same time.

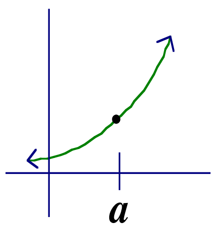
$f(x)=\dfrac{3x-2}{8-x}$		$f(x)=\dfrac{x^2+2x-15}{x+5}$		$f(x)=\dfrac{2x-1}{7x^2+4}$		$f(x)=\dfrac{(x+2)(x-3)(x-4)}{(x+2)(x-4)(x+7)}$	
vert.asym.	hole	vert.asym.	hole	vert.asym.	hole	vert.asym.	holes
$x=8$	none	none	$(-5,-8)$	none	none	$x=-7$	$(-2,-1)$ $\left(4,\dfrac{1}{11}\right)$

Continuity

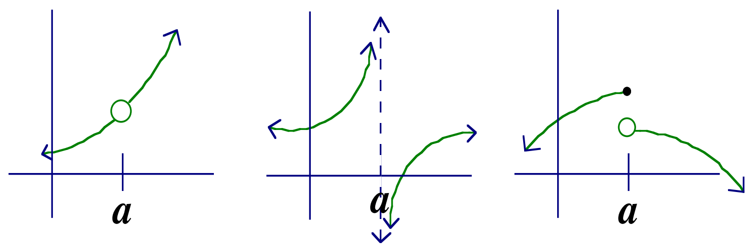
Continuous functions have **no breaks in them**.
Discontinuous functions have **breaks in them (Asymptotes or Holes / Open Circles).**

** To check for continuity at "a", you must check left hand limits $\lim\limits_{x\to -a^-} f(x)$ and right hand limits $\lim\limits_{x\to -a^+} f(x)$ as well as the value of the function at that point $f(a)$. If all three are equal then the function is continuous at a.

If $f(a)=\lim\limits_{x\to -a^-} f(x)=\lim\limits_{x\to -a^+} f(x)$ then the function is **continuous at a**.
If **$f(a)$ is not equal to either one-sided limit**, then **the function is not continuous (discontinuous) at a.**



Continuous at a



Discontinuous at a

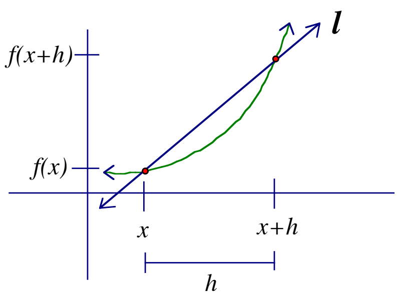
Derivative by Definition

Derivative at all points

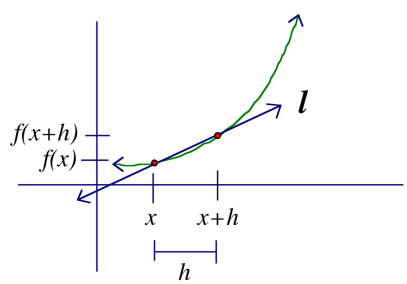
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Derivative at the point $(a, f(a))$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$



Line l is a secant line



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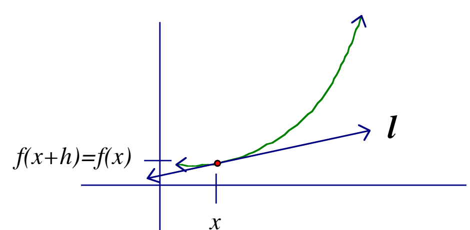


Line l is a tangent line

slope of secant line $l=\dfrac{f(x+h)-f(x)}{x+h-x}$

$\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ means that the distance h is approaching 0 and the points get closer to each other and the two points become the same point and line l is now a tangent line.

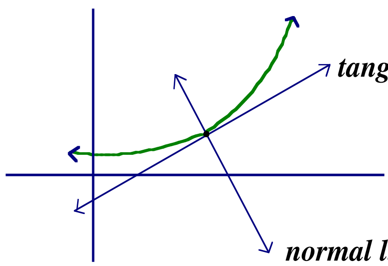
The derivative of a function finds the slope of the tangent line!

EX #1: $f(x)=3x^2$ Find $f'(x)$ Use $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{3(x+h)^2-3x^2}{h} \quad =\lim_{h\to 0}\frac{3x^2+6xh+3h^2-3x^2}{h} \quad =\lim_{h\to 0}\frac{6xh+3h^2}{h} \quad =\lim_{h\to 0}6x+3h \quad =6x$$

EX #2: $f(x)=4x^3$ Find $f'(2)$ Use $f'(a)=\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

$$f'(x)=\lim_{h\to 0}\frac{4(2+h)^3-32}{h} \quad =\lim_{h\to 0}\frac{32+48h+24h^2+4h^3-32}{h} \quad =\lim_{h\to 0}48+24h+4h^2 \quad =48$$



The **derivative finds the slope of the tangent line**.

The **normal line is perpendicular to the tangent line**.

EX #3: $f(x)=5x^2$ Find equation of the tangent line and normal line at $x=3$.

$$f'(3)=\lim_{h\to 0}\frac{5(3+h)^2-45}{h} \quad =\lim_{h\to 0}\frac{45+30h+5h^2-45}{h} \quad =\lim_{h\to 0}30+5h \quad =30 \ \text{(slope of the tangent line)}$$

Equation of a Line (point - slope form): $y-y_1=m(x-x_1)$

$f(3)=45$ and $f'(3)=30$

Equation of the tangent line: $y-45=30(x-3)$

Equation of the normal line: $y-45=\dfrac{-1}{30}(x-3)$

Questions from AP Test

When you see these problems, you need to take a derivative of the given equation.

EX#1: $\lim\limits_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h}=\cos x$

Equation: $\sin x$ Derivative: $\cos x$

EX#2: $\lim\limits_{h\to 0}\dfrac{3(x+h)^4-3x^4}{h}=12x^3$

Equation: $3x^4$ Derivative: $12x^3$

When you see these problems, you need to take a derivative of the given equation and plug in #.

EX#3: $\lim\limits_{h\to 0}\dfrac{5(2+h)^3-40}{h}=60$

Equation: $5x^3$

Derivative: $15x^2$ Derivative at $x=2$: 60

EX#4: $\lim\limits_{h\to 0}\dfrac{(1+h)^4-1}{h}=4$

Equation: x^4

Derivative: $4x^3$ Derivative at $x=1$: 4

Derivative Formulas

***Power Rule** $y = x^n$ $y' = nx^{n-1}$

EX#1: $y = 2x^5$ $y' = 10x^4$

EX#2: $y = \frac{5}{x} \Rightarrow y = 5x^{-1}$ $y' = -5x^{-2} \Rightarrow y' = \frac{-5}{x^2}$

***Product Rule** $y = f(x) \cdot g(x)$ $y' = f'(x)g(x) + g'(x)f(x)$

EX #1: $y = x^2 \sin x$ $y' = (2x) \cdot (\sin x) + (\cos x) \cdot (x^2) = 2x \sin x + x^2 \cos x$

***Quotient Rule** $y = \frac{f(x)}{g(x)}$ $y' = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$

EX #1: $y = \frac{\sin x}{x^3}$ $y' = \frac{\cos x \cdot x^3 - 3x^2 \sin x}{x^6} = \frac{x \cos x - 3 \sin x}{x^4}$

***Chain Rule** $y = (f(x))^n$ **OR** $y = f(g(x))$

$y' = n(f(x))^{n-1} \cdot f'(x)$ $y' = f'(g(x)) \cdot g'(x)$

EX #1: $y = (x^2 + 1)^3$ $y' = 3(x^2 + 1)^2 \cdot 2x = 6x(x^2 + 1)^2$

***Implicit Differentiation**- function in terms of x's and y's $\left(\text{must write } \frac{dy}{dx} \text{ everytime you take a deriv. of } y\right)$

EX#1: $x^2 y + y^3 + x^2 = 5$

derivative $\Rightarrow 2x \cdot y + \frac{dy}{dx} x^2 + 3y^2 \frac{dy}{dx} + 2x = 0$ Now solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} x^2 + 3y^2 \frac{dy}{dx} = -2x - 2xy \quad \Rightarrow \quad \frac{dy}{dx}(x^2 + 3y^2) = -2x - 2xy \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x - 2xy}{x^2 + 3y^2}$$

***Trig. Functions** (Take the derivative of the trig. function times the derivative of the angle)

Function	*Derivative*	*Example*
$\sin x$	$\cos x$	$\frac{d}{dx}\sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2)$
$\cos x$	$-\sin x$	$\frac{d}{dx}\cos^2(3x^3) = 2(\cos(3x^3)) \cdot -\sin(3x^3) \cdot 9x^2 = -18x^2 \cos(3x^3)\sin(3x^3)$
$\tan x$	$\sec^2 x$	$\frac{d}{dx}\tan(25x) = \sec^2(25x) \cdot 25 = 25\sec^2(25x)$
$\csc x$	$-\csc x \cot x$	$\frac{d}{dx}\csc(3x^4) = -\csc(3x^4)\cot(3x^4) \cdot 12x^3 = -12x^3 \csc(3x^4)\cot(3x^4)$
$\sec x$	$\sec x \tan x$	$\frac{d}{dx}\sec(\sin x) = \sec(\sin x)\tan(\sin x) \cdot \cos x$
$\cot x$	$-\csc^2 x$	$\frac{d}{dx}\cot(x^5) = -\csc^2(x^5) \cdot 5x^4 = -5x^4 \csc^2(x^5)$

***Natural Log** $y = ln\left(f(x)\right)$ $y' = \dfrac{1}{f(x)} \cdot f'(x)$

EX#1: $y = ln(x^2+1)$ $y' = \dfrac{1}{x^2+1} \cdot 2x = \dfrac{2x}{x^2+1}$ **EX#2:** $y = ln(\sin x)$ $y' = \dfrac{1}{\sin x} \cdot \cos x = \cot x$

EX#3: $y = log\ x^2$ change of base $\Rightarrow$ $y = \dfrac{ln\,x^2}{ln10}$ $y' = \dfrac{1}{ln10} \cdot \dfrac{1}{x^2} \cdot 2x = \dfrac{2}{x\,ln10}$

***Constant$^{\textbf{Variable}}$** $y = a^{f(x)}$ $y' = a^{f(x)} \cdot f'(x) \cdot ln\,a$

(3 steps: itself, derivative of exponent, ln of base)

EX#1: $y = 2^x$ $y' = 2^x \cdot 1 \cdot ln2$

EX#2: $y = 3^{x^2}$ $y' = 3^{x^2} \cdot 2x \cdot ln3$

EX#3: $y = e^{5x}$ $y' = e^{5x} \cdot 5 \cdot ln\,e = 5e^{5x}$

***Variable$^{\textbf{Variable}}$** $y = f(x)^{g(x)}$

(take **ln** of both sides)

$\ln y = g(x)\ln f(x)$ $\dfrac{1}{y}\dfrac{dy}{dx} = g'(x)\ln f(x) + \dfrac{f'(x)}{f(x)}g(x)$ $\dfrac{dy}{dx} = f(x)^{g(x)}\left(g'(x)\ln f(x) + \dfrac{f'(x)}{f(x)}g(x)\right)$

EX#1: $y = x^{\sin x}$ (take **ln** of both sides then take derivative)

$ln\,y = \sin x \cdot ln\,x$ $\dfrac{1}{y}\dfrac{dy}{dx} = \cos x \cdot ln\,x + \dfrac{1}{x} \cdot \sin x$ $\dfrac{dy}{dx} = x^{\sin x}\left[\cos x \cdot \ln x + \dfrac{\sin x}{x}\right]$

***Variable$^{\textbf{Variable}}$** $y = f(x)^{g(x)}$ (**alternate way**)

Need to change $y = f(x)^{g(x)}$ to $y = e^{ln\,f(x)^{g(x)}}$ $\Rightarrow$ $y = e^{g(x) ln\,f(x)}$ then take derivative.

EX#1: $y = x^{\sin x}$ $\Rightarrow$ $y = e^{\sin x\,ln x}$

$$y' = e^{\sin x\,ln x} \cdot \left[\cos x \cdot ln\,x + \frac{1}{x}\sin x\right] = x^{\sin x}\left[\cos x \cdot ln\,x + \frac{\sin x}{x}\right]$$

***Inverse Trig. Functions**

$y = \arcsin f(x)$ $y' = \dfrac{1}{\sqrt{1-\left(f(x)\right)^2}} \cdot f'(x)$

$y = \arctan f(x)$ $y' = \dfrac{1}{1+\left(f(x)\right)^2} \cdot f'(x)$

$y = \operatorname{arc\,sec} f(x)$ $y' = \dfrac{1}{|f(x)|\sqrt{\left(f(x)\right)^2-1}} \cdot f'(x)$

EX#1: $y = \arcsin x^4$

$y' = \dfrac{1}{\sqrt{1-x^8}} \cdot 4x^3$

$y' = \dfrac{4x^3}{\sqrt{1-x^8}}$

EX#2: $y = \arctan 2x^3$

$y' = \dfrac{1}{1+4x^6} \cdot 6x^2$

$y' = \dfrac{6x^2}{1+4x^6}$

EX#3: $y = \operatorname{arc\,sec} e^x$

$y' = \dfrac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x$

$y' = \dfrac{1}{\sqrt{e^{2x}-1}}$

Related Rates

We take derivatives with respect to t which allows us to find velocity. Here is how you take a derivative with respect to t:

derivative of x is $\frac{dx}{dt}$, derivative of y^2 is $2y\frac{dy}{dt}$, derivative of r^3 is $3r^2\frac{dr}{dt}$, derivative of t^2 is $2t\frac{dt}{dt}=2t$

V means volume ; $\frac{dV}{dt}$ means rate of change of volume (how fast the volume is changing)

r means radius ; $\frac{dr}{dt}$ means rate of change of radius (how fast the radius is changing)

$\frac{dx}{dt}$ is how fast x is changing; $\frac{dy}{dt}$ is how fast y is changing

Volume of a sphere

$$V=\frac{4}{3}\pi r^3$$
$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

Surface Area of a sphere

$$A=4\pi r^2$$
$$\frac{dA}{dt}=8\pi r\frac{dr}{dt}$$

Area of a circle

$$A=\pi r^2$$
$$\frac{dA}{dt}=2\pi r\frac{dr}{dt}$$

Circumference of a circle

$$C=2\pi r$$
$$\frac{dC}{dt}=2\pi\frac{dr}{dt}$$

Volume of a cylinder

$$V=\pi r^2 h$$

r is not a variable in a cylinder because its' value is always the same

Volume of a cone

$$V=\frac{1}{3}\pi r^2 h \quad \text{use } \frac{r}{h}=$$

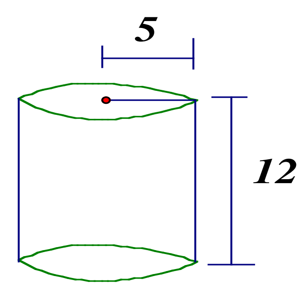
Due to similar triangles, the ratio of the radius to the height is always the same. Replace r or h depending on what you are looking for.

EX #1 : **The radius of a spherical balloon is increasing at the rate of 4 *ft / min*. How fast is the surface area of the balloon changing when the radius is 3 *ft.*?**

$$A=4\pi r^2 \quad\Rightarrow\quad \frac{dA}{dt}=8\pi r\frac{dr}{dt} \quad\Rightarrow\quad \frac{dA}{dt}=8\pi\cdot 3\cdot 4 \quad\Rightarrow\quad \frac{dA}{dt}=96\pi\ ft^2/\text{min}.$$

The surface area of the balloon is increasing at $96\pi\ ft^2/\text{min}$.

EX #2 : **Water is poured into a cylinder with radius 5 at the rate of 10 in^3/s. How fast is the height of the water changing when the height is 6 *in*?**

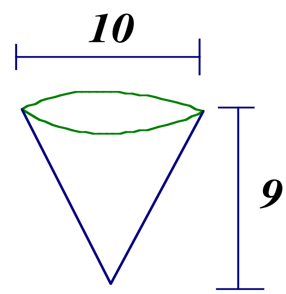


$$V=\pi r^2 h \qquad r=5 \qquad V=25\pi h$$

$$\frac{dV}{dt}=25\pi\frac{dh}{dt} \quad\Rightarrow\quad 10=25\pi\frac{dh}{dt} \quad\Rightarrow\quad \frac{dh}{dt}=\frac{2}{5\pi}\ in/s$$

The height is increasing at 0.127324 in/s.

EX #3: **Water is leaking out of a cone with diameter 10 inches and height 9 inches at the rate of 7 in^3/s. How fast is the height of the water changing when the height is 5 in?**



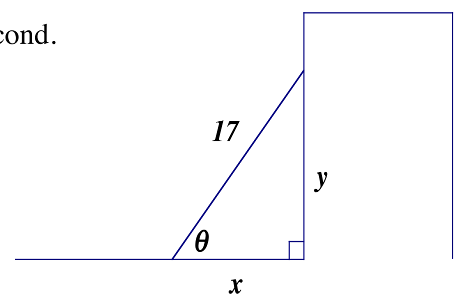
$$\frac{r}{h}=\frac{5}{9} \qquad r=\frac{5h}{9}$$

$$V=\frac{1}{3}\pi r^2 h \;\Rightarrow\; V=\frac{1}{3}\pi\left(\frac{5h}{9}\right)^2 h \;\Rightarrow\; V=\frac{25}{243}\pi h^3$$

$$\frac{dV}{dt}=\frac{25}{81}\pi h^2\frac{dh}{dt} \;\Rightarrow\; -7=\frac{25}{81}\pi 5^2\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt}=\frac{-567}{625\pi}\ in/s$$

The height is decreasing at $-0.288771\ in/s$

EX #4: **A 17 foot ladder is leaning against the wall of a house. The base of the ladder is pulled away at 3 ft. per second.**



a) How fast is the ladder sliding down the wall when the base of the ladder is 8 ft. from the wall?

$x^2+y^2=17^2 \;\Rightarrow\; y=15$ when $x=8$

$$2x\cdot\frac{dx}{dt}+2y\cdot\frac{dy}{dt}=0 \;\Rightarrow\; 2\cdot8\cdot3+2\cdot15\cdot\frac{dy}{dt}=0 \;\Rightarrow\; \frac{dy}{dt}=\frac{-8}{5}\ \text{ft. per second.}$$

b) How fast is the area of the triangle formed changing at this time?

$$A=\frac{1}{2}x\cdot y \;\Rightarrow\; \frac{dA}{dt}=\frac{1}{2}\cdot\frac{dx}{dt}\cdot y+\frac{dy}{dt}\cdot\frac{1}{2}\cdot x \;\Rightarrow\; \frac{dA}{dt}=\frac{1}{2}\cdot3\cdot15+\frac{-8}{5}\cdot\frac{1}{2}\cdot8$$

$$\frac{dA}{dt}=\frac{161}{10}\ \text{ft}^2\text{. per second.}$$

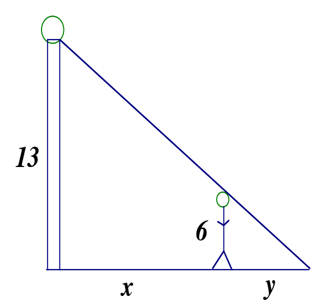
c) How fast is the angle between the bottom of the ladder and the floor changing at this time?

$$\sin\theta=\frac{y}{17} \;\Rightarrow\; \cos\theta\cdot\frac{d\theta}{dt}=\frac{1}{17}\cdot\frac{dy}{dt} \;\Rightarrow\; \frac{8}{17}\cdot\frac{d\theta}{dt}=\frac{1}{17}\cdot\frac{-8}{5}$$

$$\frac{d\theta}{dt}=\frac{-1}{5}\ \text{radians per second}$$

EX #5: **A person 6 ft. tall walks directly away from a streetlight that is 13 feet above the ground. The person is walking away from the light at a constant rate of 2 feet per second.**

a) At what rate, in feet per second, is the length of the shadow changing?



$$\frac{dx}{dt}=2\ (\text{speed of the man walking}) \qquad \frac{dy}{dt}=?\ (\text{speed of the length of shadow})$$

Use similar triangles: $\dfrac{13}{x+y}=\dfrac{6}{y} \;\Rightarrow\; 13y=6x+6y \;\Rightarrow\; 7y=6x$

$$y=\frac{6}{7}x \;\Rightarrow\; \frac{dy}{dt}=\frac{6}{7}\cdot\frac{dx}{dt} \;\Rightarrow\; \frac{dy}{dt}=\frac{6}{7}\cdot2 \;\Rightarrow\; \frac{dy}{dt}=\frac{12}{7}\ \text{feet per second}$$

b) At what rate, in feet per second, is the tip of the shadow changing?

Tip of shadow is $x+y$, so speed of tip is $\dfrac{dx}{dt}+\dfrac{dy}{dt}=2+\dfrac{12}{7}=\dfrac{26}{7}$ feet per second

Properties of Derivatives

Derivative is a **rate of change**; it finds the change in y over the change in x, $\frac{dy}{dx}$, which is slope.

1st derivative $\Rightarrow$ max. and min., increasing and decreasing, slope of the tangent line to the curve, and velocity.

2nd derivative $\Rightarrow$ inflection points, concavity, and acceleration.

Slope of the tangent line to the curve

EX #1: Given $f(x) = 3x^2 - 10x$ Find equation of the tangent line and normal line at $x = 4$.

$f(x) = 3x^2 - 10x \qquad f'(x) = 6x - 10$

$f(4) = 8 \qquad f'(4) = 14$

Equation of a Line (point - slope form): $y - y_1 = m(x - x_1)$

Equation of the tangent line: $y - 8 = 14(x - 4)$

Equation of the normal line: $y - 8 = \frac{-1}{14}(x - 4)$

Properties of First Derivative

increasing: slopes of tangent lines are positive $f'(x) > 0$

decreasing: slopes of tangent lines are negative $f'(x) < 0$

maximum point: Slopes switch from positive to negative. (found by setting $f'(x) = 0$)

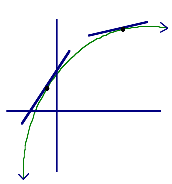
minimum point: Slopes switch from negative to positive. (found by setting $f'(x) = 0$)

Properties of Second Derivative

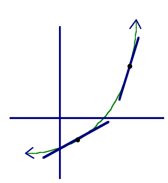
concave up: slopes of tangent lines are increasing. $f''(x) > 0$

concave down: slopes of tangent lines are decreasing. $f''(x) < 0$

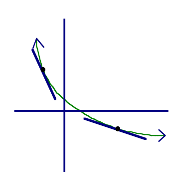
inflection points: points where the graph switches concavity. (found by setting $f''(x) = 0$)
slopes of tangent line switch from increasing to decreasing or vice versa.



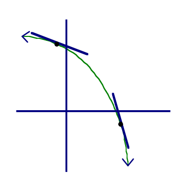
increasing/concave down
slopes are positive and decreasing



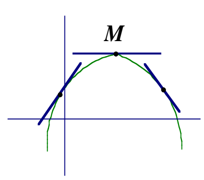
increasing/concave up
slopes are positive and increasing



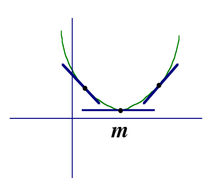
decreasing/concave up
slopes are negative and increasing



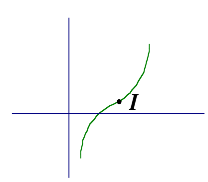
decreasing/concave down
slopes are negative and decreasing



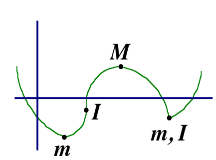
M is a Maximum; slopes switch from positive to negative



m is a Minimum; slopes switch from negative to positive



I is an inflection point; slopes switch from decreasing to increasing



M is a Maximum; m is a minimum; I is an inflection point

Application of Derivatives

To find rel. max., rel. min., where the graph is increasing and decreasing, we set the first derivative = 0

EX#1: $y = 2x^3 - 3x^2 - 36x + 2$

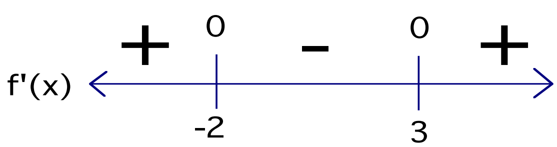
$y' = 6x^2 - 6x - 36$

$0 = 6(x^2 - x - 6)$

$0 = 6(x-3)(x+2)$

$x = 3$ and $x = -2$

1) Plug #'s in each interval on the number line into first derivative.
   $f'(x) > 0$ means graph is increasing on that interval.
   $f'(x) < 0$ means graph is decreasing on that interval.
   $f'(x)$ switches from $+$ to $-$, then point is a relative maximum.
   $f'(x)$ switches from $-$ to $+$, then point is a relative minimum.
2) To find the Y value of max. and min. plug into original equation.



max	min	increasing	decreasing
$(-2, 46)$	$(3, -79)$	$(-\infty, -2)\,(3, \infty)$	$(-2, 3)$

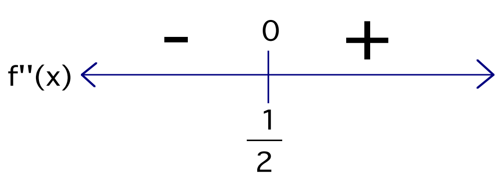
To find inflection points, where the graph is concave up and concave down, we set the second derivative = 0

$y'' = 12x - 6$

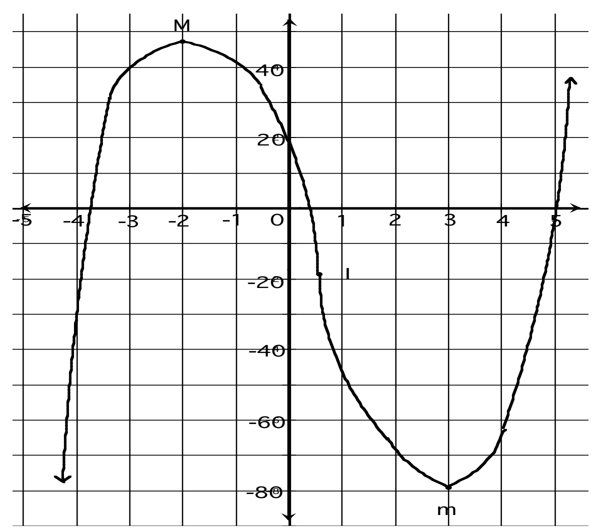
$0 = 6(2x - 1)$

$x = \frac{1}{2}$

1) Plug #'s in each interval on the number line into second derivative.
   $f''(x) > 0$ means graph is concave up on that interval.
   $f''(x) < 0$ means graph is concave down on that interval.
   An inflection point occurs at the points where $f''(x)$ switches from $+$ to $-$ or from $-$ to $+$.
2) To find the Y value of inf. pt. plug into original equation.



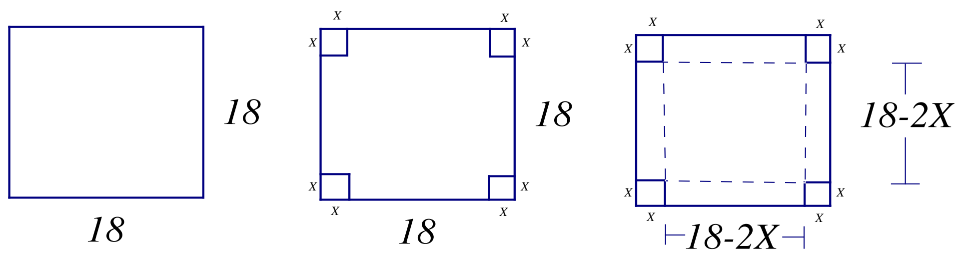
inflection pt.	concave down	concave up
$\left(\frac{1}{2}, \frac{-33}{2}\right)$	$\left(-\infty, \frac{1}{2}\right)$	$\left(\frac{1}{2}, \infty\right)$



Optimization Problems

1) Draw and label picture.
2) Write equation based on fact given and write equation for what you need to maximize or minimize.
3) Plug in fact equation into the equation you want to maximize or minimize.
4) Take derivative and set equal to zero.
5) Find remaining information.

EX#1 : An open box of maximum volume is to be made from a square piece of material, 18 inches on a side, by cutting equal squares from the corners and turning up the sides. How much should you cut off from the corners? What is the maximum volume of your box?



$V = (18-2x)^2 \cdot x$

$V = 4x^3 - 72x^2 + 324x$

$V' = 12x^2 - 144x + 324 = 0$

$V' = 12\left(x^2 - 12x + 27\right) = 0$

$V' = 12\left(x-9)(x-3\right) = 0$

$\underline{x=3}, \; x = 9$

Cutting off 9 makes no sense (minimum box). We need to cut off 3 inches to have a box with maximum volume. The maximum volume is $V = (18 - 2\cdot 3)^2 \cdot 3$ $\quad \underline{V = 432\ in^2}$

EX#2 : A farmer plans to fence a rectangular pasture adjacent to a river. The farmer has 84 feet of fence in which to enclose the pasture. What dimensions should be used so that the enclosed area will be a maximum? What is the maximum Area?

$P = 2y + x$

$84 = 2y + x$

$x = 84 - 2y$

$A = x \cdot y$

$A = \left(84 - 2y\right) \cdot y$

$A = 84y - 2y^2$

$A' = 84 - 4y = 0$

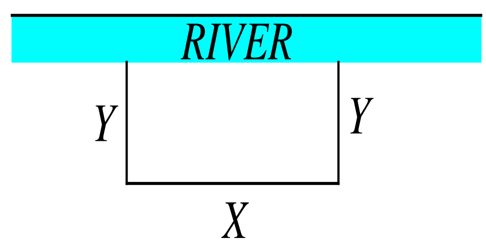
$\underline{y = 21}$

$x = 84 - 2\cdot 21$

$\underline{x = 42}$

$A = 42 \cdot 21$

$\underline{A = 882\ ft^2}$



EX#3 : A farmer plans to fence two equal rectangular pasture adjacent to a river. The farmer has 120 feet of fence in which to enclose the pastures. What dimensions should be used so that the enclosed area will be a maximum? What is the maximum Area?

$120 = 2x + 3y$

$120 - 3y = 2x$

$60 - \dfrac{3}{2}y = x$

$A = 2x \cdot y$

$A = 2\left(60 - \dfrac{3}{2}y\right) \cdot y$

$A = 120y - 3y^2$

$A' = 120 - 6y = 0$

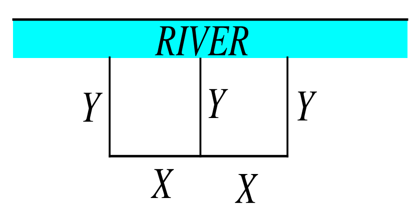
$\underline{y = 20}$

$x = 60 - \dfrac{3}{2} \cdot 20$

$\underline{x = 30}$

$A = 2 \cdot 30 \cdot 20$

$\underline{A = 1200\ ft^2}$



EX#4 : A crate, open at the top, has vertical sides, a square bottom and a volume of 500 ft^3. What dimensions give us minimum surface area? What is the surface area?

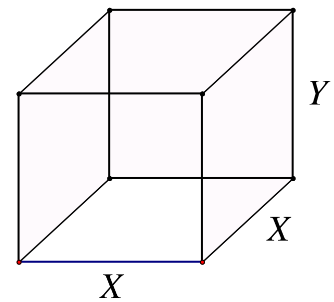
$V = x^2 \cdot y \qquad A = x^2 + 4 \cdot x \cdot y \qquad A' = 2x - \dfrac{2000}{x^2} = 0 \qquad A = 10^2 + 4 \cdot 10 \cdot 5$

$500 = x^2 \cdot y \qquad A = x^2 + 4 \cdot x \cdot \dfrac{500}{x^2} \qquad A' = \dfrac{2x^3 - 2000}{x^2} = 0 \qquad \underline{A = 300\ ft^2}$

$\dfrac{500}{x^2} = y \qquad A = x^2 + \dfrac{2000}{x} \qquad 2x^3 - 2000 = 0$

$\underline{x = 10}$

$y = \dfrac{500}{10^2} \qquad \underline{y = 5}$



EX#5 : A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{18 - x^2}$. What length and width should the rectangle have so that its area is a maximum?

$y = \sqrt{18 - x^2} \qquad A = 2 \cdot x \cdot y \qquad A' = 2 \cdot \left(18 - x^2\right)^{1/2} + \dfrac{1}{2}\left(18 - x^2\right)^{-1/2}(-2x) \cdot 2x \qquad A = 2 \cdot 3 \cdot 3$

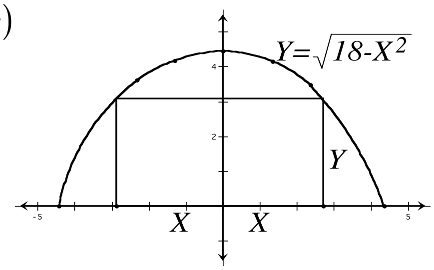
$A = 2 \cdot x \cdot \sqrt{18 - x^2} \qquad A' = 2 \cdot \left(18 - x^2\right)^{1/2} - \dfrac{2x^2}{\left(18 - x^2\right)^{1/2}} \qquad \underline{A = 18}$

$A' = \dfrac{2 \cdot \left(18 - x^2\right) - 2x^2}{\left(18 - x^2\right)^{1/2}} \quad (Bowtie)$

$A' = \dfrac{36 - 4x^2}{\left(18 - x^2\right)^{1/2}} = 0$

$y = \sqrt{18 - 3^2} \qquad 36 - 4x^2 = 0$

$\underline{y = 3} \qquad \underline{x = 3}$



Integration Formulas

***Integral of a constant** $\int a\,dx = ax + C$ **EX#1:** $\int 5\,dx = 5x + C$ **EX#2:** $\int \pi\,dx = \pi x + C$

***Polynomials** $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ **EX#1:** $\int x^3 + 5x^2 - 8x\,dx = \dfrac{x^4}{4} + \dfrac{5x^3}{3} - 4x^2 + C$

***Fractions** (Bring up denominator, then take integral)

EX#1: $\int \dfrac{1}{x^4}\,dx \Rightarrow \int x^{-4}\,dx = \dfrac{x^{-3}}{-3} + C = \dfrac{-1}{3x^3} + C$

***Constant$^{\text{Variable}}$** $\int a^x\,dx = \dfrac{a^x}{1 \cdot \ln a} + C$ **(3 steps: itself, divided by deriv. of exponent, divided by ln of base)**

EX#1: $\int 5^x\,dx = \dfrac{5^x}{\ln 5} + C$ **EX#2:** $\int 3^{2x}\,dx = \dfrac{3^{2x}}{2\ln 3} + C$

EX#3: $\int e^x\,dx = \dfrac{e^x}{\ln e} + C = e^x + C$ **EX#4:** $\int e^{2x}\,dx = \dfrac{e^{2x}}{2\ln e} + C = \dfrac{e^{2x}}{2} + C$

***Trig Functions** (**Always divide by derivative of the angle**)

$\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int \tan x\,dx = -\ln|\cos x| + C$

$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$ $\int \cot x\,dx = \ln|\sin x| + C$

EX#1: $\int \cos 2x\,dx = \dfrac{\sin 2x}{2} + C$ **EX#2:** $\int \sin 6x\,dx = \dfrac{-\cos 6x}{6} + C$

EX#3: $\int \tan 3x\,dx = -\dfrac{1}{3}\ln|\cos 3x| + C$ **EX#4:** $\int \sec 7x\,dx = \dfrac{1}{7}\ln|\sec 7x + \tan 7x| + C$

EX#5: $\int \csc 4x\,dx = -\dfrac{1}{4}\ln|\csc 4x + \cot 4x| + C$ **EX#6:** $\int \cot 9x\,dx = \dfrac{1}{9}\ln|\sin 9x| + C$

***Natural Log:** $\int \dfrac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$ $\Rightarrow$ top is the derivative of the bottom

EX#1: $\int \dfrac{1}{x}\,dx = \ln|x| + C$ **EX#2:** $\int \dfrac{3x^2}{x^3 - 5}\,dx = \ln|x^3 - 5| + C$

EX#3: $\int \dfrac{-\sin x}{\cos x}\,dx = \ln|\cos x| + C$ **EX#4:** $\int \dfrac{x^3}{x^4 + 1}\,dx = \dfrac{1}{4}\int \dfrac{4x^3}{x^4 + 1}\,dx = \dfrac{1}{4}\ln\left(x^4 + 1\right) + C$

***Integral (top is higher or same power than bottom)** **(Must divide bottom equation into top equation).**

EX#1: $\int \dfrac{x^2 + 2}{x^2 - 2x + 4}\,dx$ $\Rightarrow$ **Long Division** $= \int 1 + \dfrac{2x - 2}{x^2 - 2x + 4}\,dx = x + \ln\left|x^2 - 2x + 4\right| + C$

EX#2: $\int \dfrac{x^2 + 3x - 5}{x}\,dx = \int \left(x + 3 - \dfrac{5}{x}\right)dx = \dfrac{x^2}{2} + 3x - 5\ln x + C$

*Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+C \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+C \qquad \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\frac{x}{a}+C$$

Find variable v and constant a. The top MUST be the derivative of the variable v.

EX#1: $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx = \arcsin\frac{x}{3}+C$

$v = x \quad a = 3$

EX#2: $\displaystyle\int \frac{1}{16+x^2}\,dx = \frac{1}{4}\arctan\frac{x}{4}+C$

$v = x \quad a = 4$

EX#3: $\displaystyle\int \frac{1}{x\sqrt{x^2-25}}\,dx = \frac{1}{5}\operatorname{arcsec}\frac{|x|}{5}+C$

$v = x \quad a = 5$

EX#4: $\displaystyle\int \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{1}{3}\int \frac{3}{\sqrt{4-9x^2}}\,dx = \frac{1}{3}\arcsin\frac{3x}{2}+C$

$v = 3x \quad a = 2$

EX#5: $\displaystyle\int \frac{1}{9x^2+16}\,dx = \frac{1}{3}\int \frac{3}{9x^2+16}\,dx = \frac{1}{3}\cdot\frac{1}{4}\arctan\frac{3x}{4}+C$

$v = 3x \quad a = 4 \qquad \displaystyle = \frac{1}{12}\arctan\frac{3x}{4}+C$

EX#6: $\displaystyle\int \frac{6}{x\sqrt{49x^2-25}}\,dx = 6\int \frac{7}{|7x|\sqrt{49x^2-25}}\,dx$

$v = 7x \quad a = 5 \qquad \displaystyle = \frac{6}{5}\operatorname{arcsec}\frac{|7x|}{5}+C$

***Substitution** When integrating we usually let u = the part in the parenthesis, the part under the radical, the denominator, the exponent, or the angle of the trig. function.

EX#1: $\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int 2x\left(x^2+1\right)^{1/2} dx = \frac{1}{2}\int (u)^{1/2} du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{3}(x^2+1)^{3/2}+C$

$u = x^2+1$

$du = 2xdx$

EX#2: $\int \frac{x^2}{\left(2x^3+5\right)^4}dx = \int x^2\left(2x^3+5\right)^{-4}dx = \frac{1}{6}\int 6x^2\left(2x^3+5\right)^{-4}dx = \frac{1}{6}\int u^{-4}du = \frac{1}{6}\cdot\frac{u^{-3}}{-3}+C$

$u = 2x^3+5 \qquad du = 6x^2dx$ $\qquad = \frac{-1}{18\left(2x^3+5\right)^3}+C$

EX#3: $\int x\sqrt{x+1}\,dx = \int (u-1)\,u^{1/2}du = \int (u^{3/2}-u^{1/2})du = \frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}+C$

$u = x+1 \qquad x = u-1$ $\qquad = \frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+C$

$du = dx$

EX#4: $\int x\cos x^2\,dx = \frac{1}{2}\int 2x\cos x^2 dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\cdot\sin u+C = \frac{1}{2}\sin x^2+C$

$u = x^2$

$du = 2x\,dx$

EX#5: $\int_0^1 2x^2\left(2x^3+1\right)^4 dx = \frac{1}{3}\int_1^3 u^4 du = \frac{1}{3}\cdot\frac{u^5}{5}\Big|_1^3 = \frac{3^5}{15}-\frac{1}{15} = \frac{242}{15}$

$u = 2x^3+1$ You must switch everything from x to u. Including the #'s.

$du = 6x^2dx \qquad u = 2(0)^3+1 = 1 \qquad u = 2(1)^3+1 = 3$

Properties of Logarithms

logarithmic form $\Leftrightarrow$ **exponential form**: $y = \ln x \Leftrightarrow e^y = x$

Log Laws: $y = \ln x^3 \Leftrightarrow y = 3\ln x \qquad \ln x + \ln y = \ln xy \qquad \ln x - \ln y = \ln\left(\frac{x}{y}\right)$

$\ln 8 = \ln 2^3 = 3\ln 2 \qquad \ln 2+\ln 5 = \ln 10 \qquad \ln 7-\ln 2 = \ln\left(\frac{7}{2}\right)$

Change of Base Law: $y = \log_a x \Rightarrow y = \frac{\ln x}{\ln a}$ **Memorize:** $\ln e = 1 \quad \ln 1 = 0 \quad \log 10 = 1 \quad \log 1 = 0$

Fact: You can't take a ln/log of a negative # or zero.

We use logarithms to solve any problem that has a variable in the exponent.

EX#1: $e^{5x} = 24 \Rightarrow \ln e^{5x} = \ln 24 \Rightarrow 5x\ln e = \ln 24 \Rightarrow x = \frac{\ln 24}{5}$

EX#2: $\ln x = 3 \Rightarrow e^{\ln x} = e^3 \Rightarrow x = e^3$

***Newton's Method** (Used to approximate the zero of the function)

$$c-\frac{f(c)}{f'(c)}$$ where c is the approximation for the zero.

EX#1: If Newton's method is used to approximate the real root of $x^3+x-1=0$, then a first approximation $x_1=1$ would lead to a third approximation of $x_3=$

$f(x)=x^3+x-1 \qquad f'(x)=3x^2+1$

Plug in x_1 to find x_2 $\quad 1-\dfrac{f(1)}{f'(1)}=\dfrac{3}{4}$ or $0.75=x_2$ Plug in x_2 to find x_3 $\quad \dfrac{3}{4}-\dfrac{f(3/4)}{f'(3/4)}=\dfrac{59}{86}$ or $0.686=x_3$

DIFFERENTIAL EQUATIONS (Separating Variables) (used when you are given the derivative and you need to find the original equation. We separate the x's and y's and take the integral).

EX#1: Find the general solution given $\dfrac{dy}{dx}=\dfrac{x^2}{y}$

$$\frac{dy}{dx}=\frac{x^2}{y} \Rightarrow y\,dy=x^2\,dx \Rightarrow \int y\,dy=\int x^2\,dx \Rightarrow \frac{y^2}{2}=\frac{x^3}{3}+C \Rightarrow \underline{y^2=\frac{2x^3}{3}+C_1}$$

EX#2: Find the particular solution $y=f(x)$ for **EX#1** given $(3,-5)$

$$y^2=\frac{2x^3}{3}+C_1 \Rightarrow 25=18+C_1 \Rightarrow 7=C_1 \Rightarrow y^2=\frac{2x^3}{3}+7 \Rightarrow \underline{y=-\sqrt{\frac{2x^3}{3}+7}}$$

EX#3: Find the particular solution $y=f(x)$ given $\dfrac{dy}{dx}=6xy$ and $(0,5)$

$$\frac{dy}{y}=6x\,dx \Rightarrow \int\frac{dy}{y}=\int 6x\,dx \Rightarrow \ln y=3x^2+C \Rightarrow y=e^{3x^2+C} \Rightarrow y=C_1e^{3x^2} \Rightarrow 5=C_1(1) \Rightarrow \underline{y=5e^{3x^2}}$$

If the rate of growth of something is proportional to itself ($y'=ky$), then it is the growth formula.

Proof: $y'=ky \Rightarrow \dfrac{dy}{dt}=ky \Rightarrow \dfrac{dy}{y}=k\,dt \Rightarrow \displaystyle\int\frac{dy}{y}=\int k\,dt \Rightarrow \ln y=kt+C \Rightarrow$

$$e^{\ln y}=e^{kt+C} \Rightarrow y=e^{kt}\cdot e^{C} \Rightarrow y=C_1e^{kt}$$

***Average Value** (use this when you are asked to find the average of anything)

$$\frac{1}{b-a}\cdot\int_a^b f(x)\,dx$$

EX#1: Find the average value of $f(x)=x^3-4x$ from $[1,4]$

$$\text{Avg. value}=\frac{1}{4-1}\int_1^4 x^3-4x\,dx=\frac{1}{3}\left[\frac{x^4}{4}-2x^2\Big|_1^4\right]=\frac{1}{3}\left[(64-32)-\left(\frac{1}{4}-2\right)\right]=\frac{1}{3}\cdot\frac{127}{4}=\frac{127}{12}$$

*Continuity / Differentiability Problem

EX#1: $f(x)=\begin{cases} x^2 & , \ x<3 \\ 6x-9 & , \ x\ge 3 \end{cases}$ At 3 $f(x)=\begin{cases} (3)^2 & = & 9 \\ 6(3)-9 & = & 9 \end{cases}$

At 3 $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x) = f(3)=9$. Therefore $f(x)$ is continuous.

$f(x)$ is continuous iff both halves of the function have the same answer at the breaking point.

$f'(x)=\begin{cases} 2x & , \ x<3 \\ 6 & , \ x\ge 3 \end{cases}$ At 3 $f'(x)=\begin{cases} 2(3) & = & 6 \\ 6 & = & 6 \end{cases}$

At 3 both halves of the derivative = 6.
Therefore the function is differentiable.

$f(x)$ is differentiable if and only if the derivative of both halves of the function have the same answer at the breaking point.

Since both sides of $f(x)$ and $f'(x)$ agree at 3, then $f(x)$ is continuous and differentiable at $x = 3$.

*Rectilinear Motion (Position, Velocity, Acceleration Problems)

-We designate position as $x(t), y(t), or\ s(t)$.

-The derivative of position, $x'(t)=v(t) \Rightarrow$ velocity.

-The derivative of velocity, $v'(t)=a(t) \Rightarrow$ acceleration.

-We often talk about position, velocity, and acceleration when we're discussing particles moving along the x-axis or y-axis.

-A particle is at rest or is changing direction when $v(t)=0$.

-A particle is moving to the right or up when $v(t)>0$ and to the left or down when $v(t)<0$.

-To find the average velocity of a particle $\Rightarrow$ $\dfrac{1}{b-a}\displaystyle\int_a^b v(t)\,dt$

-To find the maximum or minimum acceleration of a particle set $a'(t) = 0$, then check the values on a number line to see if and how they switch signs.

-Speed is the absolute value of velocity, $|v(t)|$.

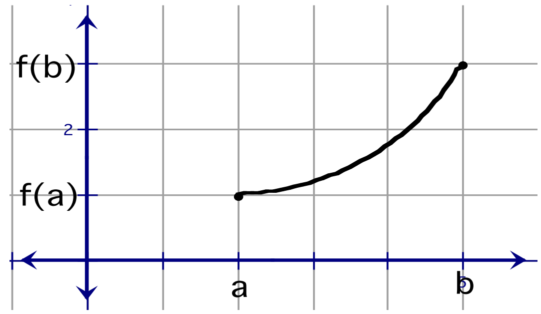
If $v(t)$ and $a(t)$ agree $\left.\begin{array}{cc} \underline{v(t)} & \underline{a(t)} \\ + & + \\ - & - \end{array}\right\}$ speed is increasing.

If $v(t)$ and $a(t)$ disagree $\left.\begin{array}{cc} \underline{v(t)} & \underline{a(t)} \\ + & - \\ - & + \end{array}\right\}$ speed is decreasing.

-Distance traveled is $\displaystyle\int_a^b |v(t)|\,dt$

*Mean - Value Theorem

(Only **applies** if the **function is continuous and differentiable**)

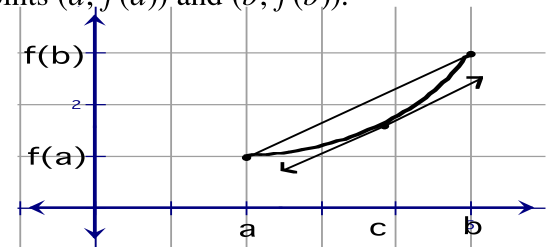


Slope of tangent line = slope of line between two points

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

According to the Mean Value Theorem, **there must be a number c** between a and b that the slope of the tangent line at c is the same as the slope between points $(a, f(a))$ and $(b, f(b))$.

The slope of secant line from a and b is the same as slope of tangent line through c.



*When given the graph of $f'(x)$, it is like you are looking at a # line.

This is the graph of $f'(x)$. Where $f'(x) = 0$ (x-int) is where the possible max. and min. are.

Signs are based on if the graph is above (increasing) or below the x - axis (decreasing).

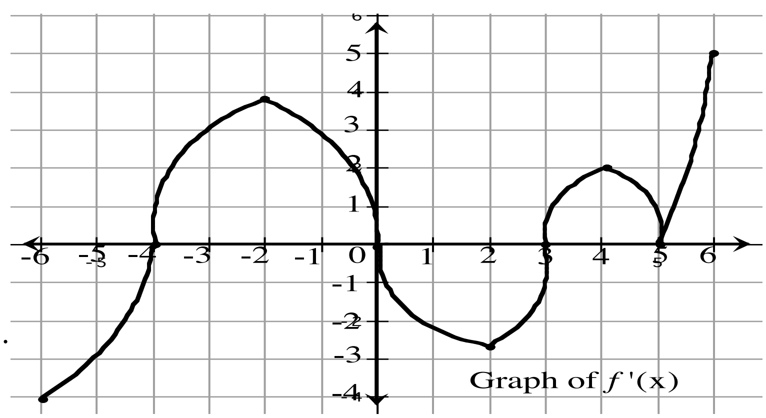
EX: Graph of $f'(x)$ from $-6 \le x \le 6$

increasing when $f'(x) > 0$

decreasing when $f'(x) < 0$

max. occurs when $f'(x)$ switches from $+$ to $-$.

min. occurs when $f'(x)$ switches from $-$ to $+$.

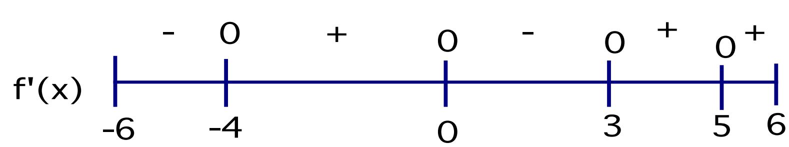


The $f''(x)$ is the slope of the tangent line of $f'(x)$.

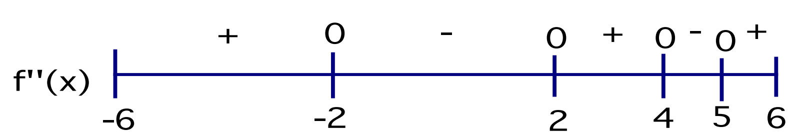
concave up when slope is positive.

concave down when slope is negative.

inf. pts. occur when slopes switch from $+$ to $-$ or $-$ to $+$.



rel.min	rel. max	increasing	decreasing
$x = -4, 3$	$x = 0$	$(-4, 0)$ $(3,6]$	$[-6, -4)$ $(0, 3)$



inf. pts.	concave up	concave down
$x = -2, 2, 4, 5$	$(-6, -2)$ $(2, 4)$ $(5, 6)$	$(-2, 2)$ $(4, 5)$

***Double - Life Formula** $\Rightarrow$ $y = C\,(2)^{t/d}$

***Half - Life Formula** $\Rightarrow$ $y = C\left(\frac{1}{2}\right)^{t/h}$

***Growth Formula** $\Rightarrow$ $y = C\,e^{kt}$ (Comes from $y' = ky$)

y = ending amount, C= intial amount, t = time, d = double-life time , h = half-life time , k = growth constant

*Trigonometric Identities

***Reciprocal Identities**

$$\frac{1}{\cos\theta} = \sec\theta$$

$$\frac{1}{\sin\theta} = \csc\theta$$

***Identities**

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

***Double Angle Formulas**

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

***Half Angle Formulas**

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

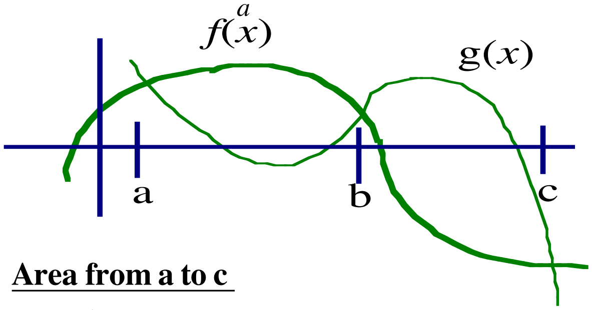
$$\cos^2 x = \frac{1+\cos 2x}{2}$$

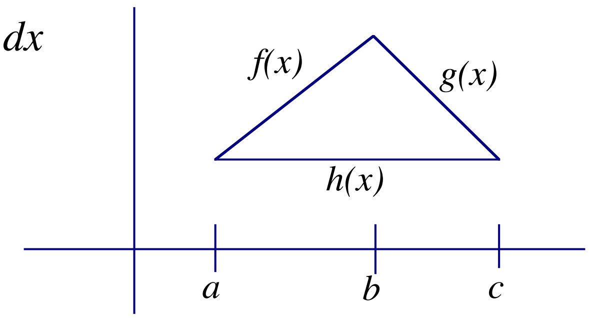
***Pythagorean Identities**

$$\sin^2 x + \cos^2 x = 1 \qquad 1+\tan^2 x = \sec^2 x \qquad 1+\cot^2 x = \csc^2 x$$

Applications of Integrals (Area, Volume, Sums)

***Area** $A = \int_a^b [\text{top equation-bottom equation}]\, dx$





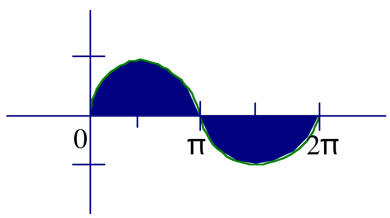
Area from a to c

$$A = \int_a^b [f(x) - g(x)]\, dx + \int_b^c [g(x) - f(x)]\, dx$$

Area from a to c

$$A = \int_a^b [f(x) - h(x)]\, dx + \int_b^c [g(x) - h(x)]\, dx$$

Find Area of $y = \sin x$ **from 0 to** 2π



From 0 to π, $\sin x$ is on top of x-axis $(y=0) \Rightarrow (\sin x - 0)$

From π to 2π, x-axis $(y=0)$ is on top of $\sin x \Rightarrow (0 - \sin x)$

$$Area = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x\, dx = -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{2\pi} = 2 + 2 = 4$$

***Volume** $V = \int A(x)\, dx$ **where** $A(x)$ **is the Area of the cross section.**

Volume if cross section rotated is a circle $(A = \pi r^2)$

$$V = \pi \int_a^b \left[(\text{top function})^2 - (\text{bottom function})^2 \right] dx$$

radius

Volume if cross section rotated is a cylinder $(A = 2\pi rh)$

$$V = 2\pi \int_a^b x [\text{top function} - \text{bottom function}]\, dx$$

radius **height**

***Volume rotated about :** **(Vertical cross section)**

the x - axis

$$\pi \int_a^b \left[(f(x))^2 - (g(x))^2 \right] dx$$

the line $y = -k$

$$\pi \int_a^b \left[(f(x) + k)^2 - (g(x) + k)^2 \right] dx$$

the line $y = m$

$$\pi \int_a^b \left[(m - g(x))^2 - (m - f(x))^2 \right] dx$$

the y - axis

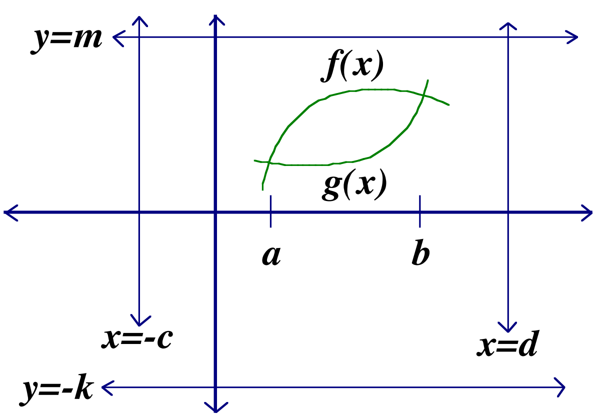
$$2\pi \int_a^b x [f(x) - g(x)]\, dx$$

the line $x = -c$

$$2\pi \int_a^b (x + c)[f(x) - g(x)]\, dx$$

the line $x = d$

$$2\pi \int_a^b (d - x)[f(x) - g(x)]\, dx$$



*Volume rotated about: (Vertical cross section)

the x - axis

$$\pi\int_0^2\left[(4)^2-\left(x^2\right)^2\right]dx$$

$$=\frac{128\pi}{5}\doteq\ 80.425$$

the y - axis

$$2\pi\int_0^2 x\left[4-x^2\right]dx$$

$$=\ 8\pi\doteq\ 25.133$$

the line $y=-2$

$$\pi\int_0^2\left[(4+2)^2-\left(x^2+2\right)^2\right]dx$$

$$=\frac{704\pi}{15}\doteq\ 147.445$$

the line $x=-3$

$$2\pi\int_0^2(x+3)\left[4-x^2\right]dx$$

$$=\ 40\pi\doteq\ 125.664$$

the line $y=5$

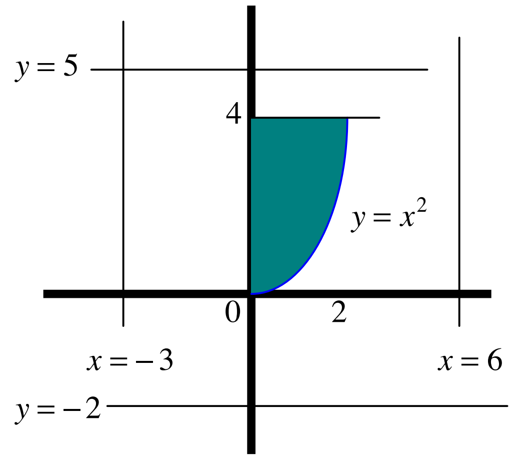
$$\pi\int_0^2\left[\left(5-x^2\right)^2-(5-4)^2\right]dx$$

$$=\frac{416\pi}{15}\doteq\ 87.127$$

the line $x=6$

$$2\pi\int_0^2(6-x)\left[4-x^2\right]dx$$

$$=\ 56\pi\doteq\ 175.929$$



*Volume rotated about: (Horizontal cross section)

the x - axis

$$2\pi\int_0^4 y\left[\sqrt{y}-0\right]dy$$

$$=\frac{128\pi}{5}\doteq\ 80.425$$

the y - axis

$$\pi\int_0^4\left[\left(\sqrt{y}\right)^2-(0)^2\right]dy$$

$$=\ 8\pi\doteq\ 25.133$$

the line $y=-2$

$$2\pi\int_0^4(y+2)\left[\sqrt{y}-0\right]dy$$

$$=\frac{704\pi}{15}\doteq\ 147.445$$

the line $x=-3$

$$\pi\int_0^4\left[\left(\sqrt{y}+3\right)^2-(0+3)^2\right]dy$$

$$=\ 40\pi\doteq\ 125.664$$

the line $y=5$

$$2\pi\int_0^4(5-y)\left[\sqrt{y}-0\right]dy$$

$$=\frac{416\pi}{15}\doteq\ 87.127$$

the line $x=6$

$$\pi\int_0^4\left[(6-0)^2-\left(6-\sqrt{y}\right)^2\right]dy$$

$$=\ 56\pi\doteq\ 175.929$$

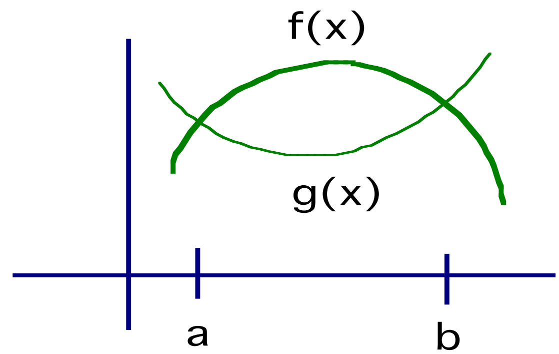
For horizontal cross sections we must switch everything from x to y.

$\underline{x}$	$\Rightarrow$	$\underline{y}$
0 to 2	$\Rightarrow$	0 to 4
$y=x^2$	$\Rightarrow$	$x=\pm\sqrt{y}$

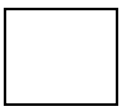
***Volume** **(Region is not rotated)** $V = \int A(x)\,dx$ **where $A(x)$ is the Area of the cross section.**

- Sometimes we will find the volume of regions that have different cross sections (not a circle or a cylinder).
- These regions are not rotated but come out at us.
- We must first find the Area of the cross section, then take it's integral.

EX#1: Let R be the region in the first quadrant below $f(x)$ and above $g(x)$ from $x = a$ to $x = b$. Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are :



Squares $\left(A = s^2\right)$

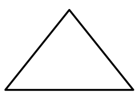


$f(x) - g(x)$

$f(x) - g(x)$

$$V = \int_a^b \left(f(x) - g(x)\right)^2 dx$$

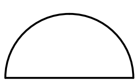
Equilateral Δ's $\left(A = \frac{s^2\sqrt{3}}{4}\right)$



$f(x) - g(x)$

$$V = \int_a^b \left(f(x) - g(x)\right)^2 \left(\frac{\sqrt{3}}{4}\right) dx \qquad V = \frac{\sqrt{3}}{4}\int_a^b \left(f(x) - g(x)\right)^2 dx$$

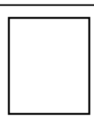
Semicircle $\left(A = \frac{\pi r^2}{2}\right)$



$f(x) - g(x) = diameter \qquad \frac{f(x)-g(x)}{2} = radius$

$$V = \int_a^b \frac{\pi}{2}\left(\frac{f(x)-g(x)}{2}\right)^2 dx \qquad V = \frac{\pi}{8}\int_a^b \left(f(x) - g(x)\right)^2 dx$$

Rectangle with $h = 5 \cdot b$ $\left(A = 5bh\right)$

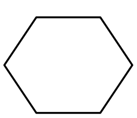


$5 \cdot \left(f(x) - g(x)\right)$

$f(x) - g(x)$

$$V = \int_a^b 5\left(f(x) - g(x)\right)\left(f(x) - g(x)\right)dx \qquad V = 5\int_a^b \left(f(x) - g(x)\right)^2 dx$$

Regular Hexagon $\left(A = \frac{1}{2}ap\right)$



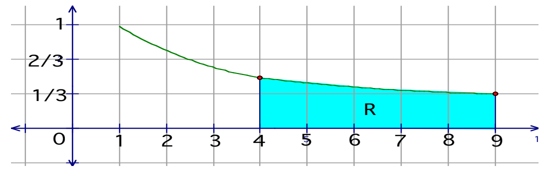
$f(x) - g(x)$

$$V = \int_a^b \frac{1}{2}\left(\frac{\tan 60^\circ}{2}\left[f(x) - g(x)\right]\right)\left(6\left(f(x) - g(x)\right)\right)dx \qquad V = \frac{3\sqrt{3}}{2}\int_a^b \left(f(x) - g(x)\right)^2 dx$$

$\frac{\tan 60^\circ}{2}\left[f(x) - g(x)\right] = apothem \qquad 6\left(f(x) - g(x)\right) = perimeter$

EX#1: Let R be the region in the first quadrant under the graph of $y=\dfrac{1}{\sqrt{x}}$ for $4 \le x \le 9$.

Find the volume of the solid whose base is the region R and whose cross sections cut by planes perpendicular to the x-axis are :



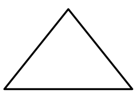
Squares$\left(A=s^2\right)$

$\dfrac{1}{\sqrt{x}}$

$$V=\int_4^9\left(\frac{1}{\sqrt{x}}\right)^2 dx=\int_4^9\frac{1}{x}\,dx=\ln x\Big|_4^9=\ln 9-\ln 4=\ln\frac{9}{4}=2\ln\frac{3}{2}\doteq 0.811$$

$\dfrac{1}{\sqrt{x}}$

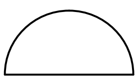
Equilateral Δ's$\left(A=\dfrac{s^2\sqrt{3}}{4}\right)$



$$V=\int_4^9\left(\frac{1}{\sqrt{x}}\right)^2\left(\frac{\sqrt{3}}{4}\right)dx=\frac{\sqrt{3}}{4}\int_4^9\frac{1}{x}\,dx=\frac{\sqrt{3}}{4}2\ln\frac{3}{2}=\frac{\sqrt{3}}{2}\ln\frac{3}{2}\doteq 0.351$$

$\dfrac{1}{\sqrt{x}}$

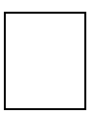
Semicircle$\left(A=\dfrac{\pi r^2}{2}\right)$



$$V=\int_4^9\frac{\pi}{2}\left(\frac{1}{2\sqrt{x}}\right)^2 dx=\int_4^9\frac{\pi}{8x}\,dx=\frac{\pi}{8}\int_4^9\frac{1}{x}\,dx=\frac{\pi}{8}\left(2\ln\frac{3}{2}\right)=\frac{\pi}{4}\ln\frac{3}{2}\doteq 0.318$$

$\dfrac{1}{\sqrt{x}}=diameter$ $\quad\dfrac{1}{2\sqrt{x}}=radius$

Rectangle with $h=5\cdot b$ $\left(A=5\cdot b\cdot h\right)$

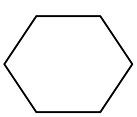


$5\cdot\dfrac{1}{\sqrt{x}}$

$$V=\int_4^9 5\left(\frac{1}{\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}}\right)dx=\int_4^9\frac{5}{x}\,dx=5\int_4^9\frac{1}{x}\,dx=5\left(2\ln\frac{3}{2}\right)=10\ln\frac{3}{2}\doteq 4.055$$

$\dfrac{1}{\sqrt{x}}$

Regular Hexagon $\left(A=\dfrac{1}{2}ap\right)$



$$V=\int_4^9\frac{1}{2}\left(\frac{\sqrt{3}}{2\sqrt{x}}\right)\left(\frac{6}{\sqrt{x}}\right)dx=\int_4^9\frac{3\sqrt{3}}{2x}\,dx=\frac{3\sqrt{3}}{2}\int_4^9\frac{1}{x}\,dx=\frac{3\sqrt{3}}{2}\left(2\ln\frac{3}{2}\right)=3\sqrt{3}\ln\frac{3}{2}\doteq 2.107$$

$\dfrac{1}{\sqrt{x}}$ $\quad\dfrac{\sqrt{3}}{2\sqrt{x}}=apothem$ $\quad\dfrac{6}{\sqrt{x}}=perimeter$

Multipliers for other figures : 30-60-90(SL): $\dfrac{\sqrt{3}}{2}$ 30-60-90(LL): $\dfrac{1}{2\sqrt{3}}$ 30-60-90(HYP): $\dfrac{\sqrt{3}}{8}$

45-45-90(LEG): $\dfrac{1}{2}$ 45-45-90(HYP): $\dfrac{1}{4}$ Regular Octagon : $2\tan 67.5°$ or $2\tan\dfrac{3\pi}{8}$

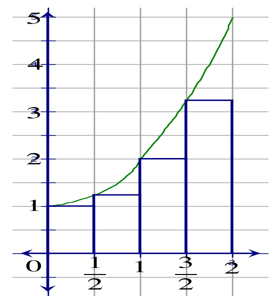
Approximating Area

We approximate Area using rectangles (left, right, and midpoint) and trapezoids.

*Riemann Sums

a) **Left edge Rectangles** $f(x)=x^2+1$ from [0, 2] using 4 subdivisions

(Find area of each rectangle and add together)

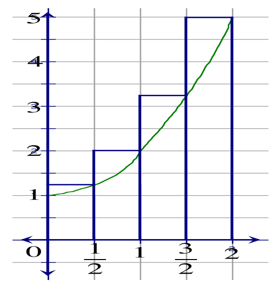


$$A = \left(\frac{1}{2}\right)\left(1+\frac{5}{4}+2+\frac{13}{4}\right)$$

Total Area= $\frac{30}{8} \doteq 3.750$

b) **Right edge Rectangles** $f(x)=x^2+1$ from [0, 2] using 4 subdivisions

(Find area of each rectangle and add together)

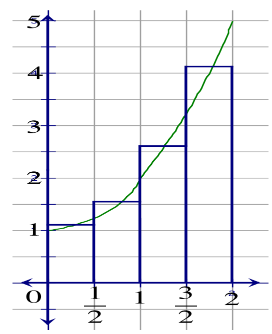


$$A = \left(\frac{1}{2}\right)\left(\frac{5}{4}+2+\frac{13}{4}+5\right)$$

Total Area = $\frac{46}{8} \doteq 5.750$

c) **Midpoint Rectangles** $f(x)=x^2+1$ from [0, 2] using 4 subdivisions

(Find area of each rectangle and add together)



$$A = \left(\frac{1}{2}\right)\left(\frac{17}{16}+\frac{25}{16}+\frac{41}{16}+\frac{65}{16}\right)$$

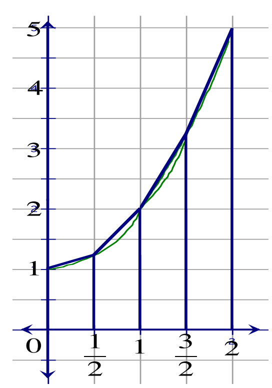
Total Area = $\frac{148}{32} \doteq 4.625$

d) **Actual Area** $= \int_0^2 \left(x^2+1\right)dx = \frac{x^3}{3}+x \Big|_0^2 = \frac{14}{3} = 4.667$

*Trapezoidal Rule (used to approximate area under a curve, using trapezoids).

$$Area \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3).....2f(x_{n-1})+f(x_n)\right]$$

where n is the number of subdivisions.



EX#1: $f(x)=x^2+1$ Approximate the area under the curve from [0, 2] using the trapezoidal rule with 4 subdivisions.

$$A = \frac{2-0}{2(4)}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+f(2)\right]$$

$$= \frac{1}{4}\left[1+2\left(\frac{5}{4}\right)+2(2)+2\left(\frac{13}{4}\right)+5\right]$$

$$= \frac{1}{4}\left[\frac{76}{4}\right] = \frac{76}{16} = 4\frac{3}{4} = 4.750$$

All you are doing is finding the area of the 4 trapezoids and adding them together!

*Approximating Area when given data only (no equation given)

To estimate the area of a plot of land, a surveyor takes several measurements. The measurements are taken every 15 feet for the 120 ft. long plot of land, where y represents the distance across the land at each 15 ft. increment.

x	0	15	30	45	60	75	90	105	120
y	58	63	72	60	62	69	61	74	67

a) Estimate using Trapezoidal Rule

$$A \doteq \frac{120-0}{2(8)}\left[f(0)+2f(15)+.....+2f(105)+f(120)\right]$$

$$A \doteq \frac{15}{2}\left[58+126+144+120+124+138+122+148+67\right]$$

$$A \doteq 7852.5$$

b) Estimate using 4 Midpoint subdivisions

$$A \doteq \frac{120-0}{4}\left[f(15)+f(45)+f(75)+f(105)\right]$$

$$A \doteq 30\left[63+60+69+74\right]$$

$$A \doteq 7980$$

c) Estimate Avg. value using Trapezoidal Rule

$$\text{Avg.Value} \doteq \frac{1}{120}(7852.5) \doteq 65.4375$$

d) What are you finding in part c?

The average distance across the land.

e) Estimate using Left Endpoint

$$A \doteq \frac{120-0}{8}\left[f(0)+f(15)+f(30)+.....+f(105)\right]$$

$$A \doteq 15\left[58+63+72+60+62+69+61+74\right]$$

$$A \doteq 7785$$

f) Estimate using Right Endpoint

$$A \doteq \frac{120-0}{8}\left[f(15)+f(30)+f(45).....+f(120)\right]$$

$$A \doteq 15\left[63+72+60+62+69+61+74+67\right]$$

$$A \doteq 7920$$

*Approximating Area when given data only (no equation given)

Unequal subdivisions: You must find each Area separately.

x	0	2	5	10
y	10	13	11	15

a) Estimate using Trapezoids $\left(A = \frac{1}{2}(b_1+b_2)h\right)$

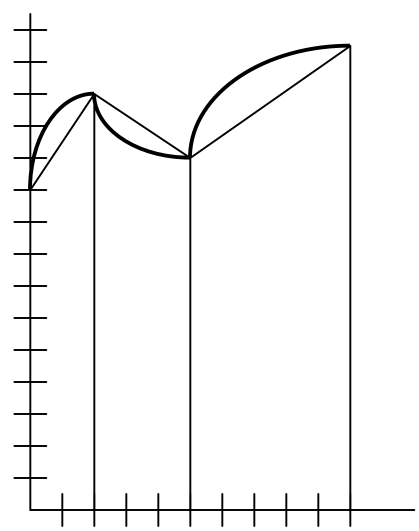
$$A \doteq \frac{1}{2}\cdot(10+13)\cdot 2 \;+\; \frac{1}{2}\cdot(13+11)\cdot 3 \;+\; \frac{1}{2}\cdot(11+15)\cdot 5 \quad \doteq 124$$

b) Estimate using Left Endpoint $(A = width \cdot left\ height)$

$$A \doteq 2(10)+3(13)+5(11) \qquad \doteq 114$$

c) Estimate using Right Endpoint $(A = width \cdot right\ height)$

$$A \doteq 2(13)+3(11)+5(15) \qquad \doteq 134$$



Trapezoids shown

1st Fundamental Theorem of Calculus

Just plug in the top # minus the bottom #.

EX#1: $\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$

EX#2: $\int_{\pi/4}^{\pi} \sin x\,dx = -\cos x\Big|_{\pi/4}^{\pi} = 1 - \frac{-\sqrt{2}}{2} = 1 + \frac{\sqrt{2}}{2}$

2nd Fundamental Theorem of Calculus (When taking the derivative of an integral)

Plug in the variable on top times its derivative minus plug in the variable on bottom times its derivative.

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$

EX#1: $\frac{d}{dx}\int_0^x t^3\,dt = x^3 \cdot 1 - 0 = x^3$

EX#2: $\frac{d}{dx}\int_{x^2}^0 t^3\,dt = 0 - x^6 \cdot 2x = -2x^7$

EX#3: $\frac{d}{dx}\int_x^{x^2} \sqrt{t^5+2}\,dt = \sqrt{x^{10}+2} \cdot 2x - \sqrt{x^5+2} \cdot 1 = 2x\sqrt{x^{10}+2} - \sqrt{x^5+2}$

Integral as an accumulator

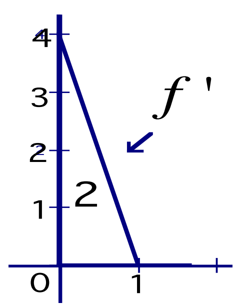
A definite integral finds the change in the equation above it.

The integral of velocity from a to b is the change in position (distance travelled) from a to b.

The integral of acceleration from 0 to 3 is the change in velocity from time 0 to time 3.

The integral of $f'(x)$ is the change in $f(x)$.

EX#1: If $f(0) = 5$ then find $f(1)$.

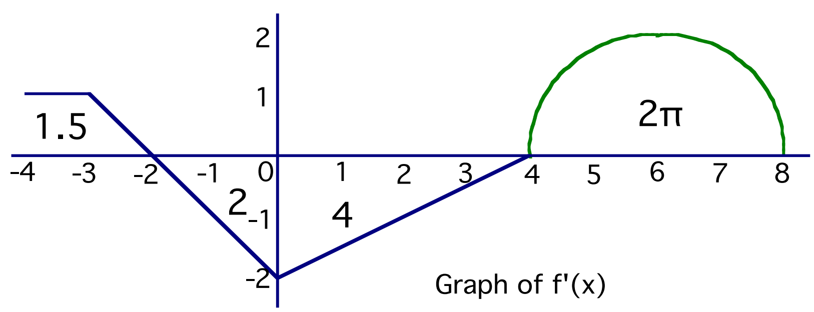


Since $\int_0^1 f'(x)\,dx = f(1) - f(0)$ it finds the change in f from 0 to 1 .

Since the area under f' from 0 to 1 = 2 this will help find $f(1)$.

$f(1) = f(0) + \int_0^1 f'(x)\,dx = 5 + 2 = 7$

EX#2: Integrals going left are negative.
Integrals going right are positive.



If $f(0) = 10$ then:

$f(-4) = 10 + \int_0^{-4} f'(x)\,dx = 10 + 2 - 1.5 = 10.5$

$f(-2) = 10 + \int_0^{-2} f'(x)\,dx = 10 + 2 = 12$

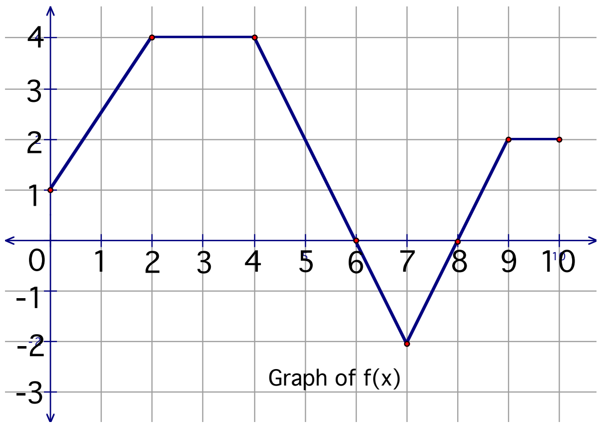
$f(4) = 10 + \int_0^4 f'(x)\,dx = 10 - 4 = 6$

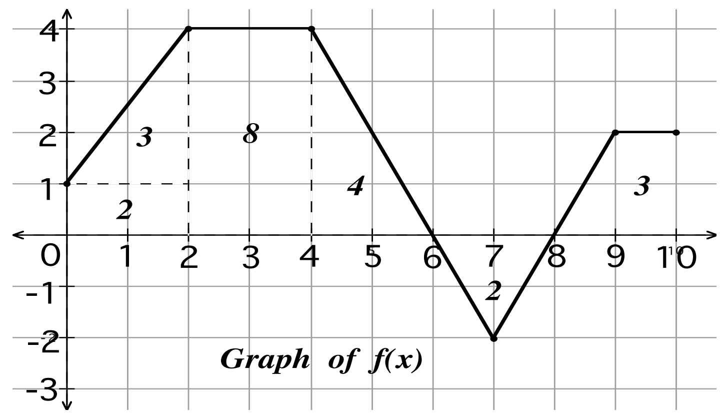
$f(8) = 10 + \int_0^8 f'(x)\,dx = 10 - 4 + 2\pi = 6 + 2\pi$

EX#3: Given $v(2) = 8$ and $a(t) = \sin\left(t^2+1\right)$ find $v(5)$.

$v(5) = v(2) + \int_2^5 a(t)\,dt = 8 + -0.02336 = 7.97663$ (The integral of acceleration finds the change in velocity)

Finding Derivatives and Integrals given a graph of *f(x)*





The derivative is the slope of each line.

$f'(1) = 3/2$

$f'(2) = DNE$

$f'(3) = 0$

$f'(5) = -2$

$f'(7) = DNE$

$f'(8) = 2$

$f'(9) = DNE$

$f'(9.5) = 0$

The integral finds the total area between *f(x)* and the *x* - axis.

$\int_0^{10} f(x)\,dx = \quad 17 - 2 + 3 = \quad 18$

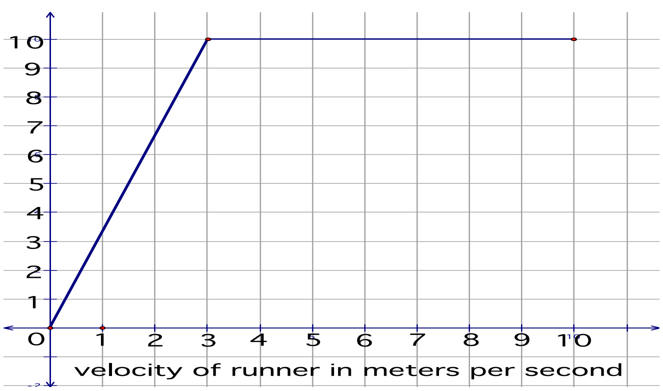
$\int_0^{10} |f(x)|\,dx = \quad 17 + 2 + 3 = \quad 22$

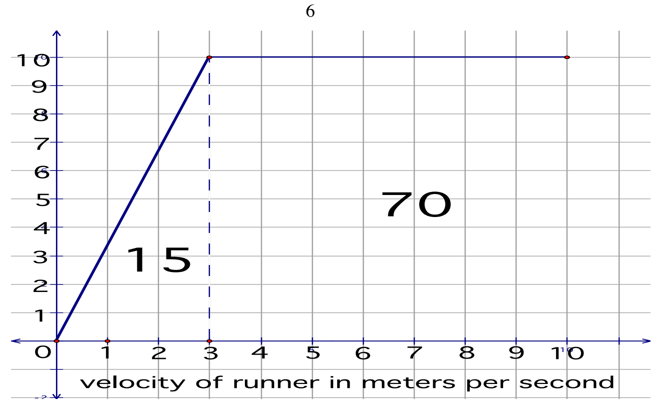
$\int_0^{10} (f(x) + 5)\,dx = \quad 68$

$\int_{10}^{8} f(x)\,dx = \quad -3$

$\int_8^{2} f(x)\,dx = \quad 2 - 12 = \quad -10$

$\int_6^{10} f(x)\,dx = \quad 1$





EX#1 :

Find the velocity of the runner at $t = 2$ and $t = 7$ seconds.

$v(2) = \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$ $\qquad v(7) = \quad 10$

Find the acceleration of the runner at $t = 2$ and $t = 7$ seconds

Since $v'(t) = a(t)$, you find acceleration by finding the derivative (slope) of velocity.

$a(2) = \dfrac{10}{3}$ $\qquad a(7) = \quad 0$

Find the distance travelled by the runner from $t = 0$ and $t = 10$ seconds

Distance travelled = $\int_0^{10} |v(t)|\,dt$ $\qquad \int_0^{10} |v(t)|\,dt = \quad 85$

EX#2 :

Given $x(0) = 45$ find $x(3)$ and $x(10)$.

$x(3) = \quad 45 + \int_0^{3} v(t)\,dt =$

$45 + 15 = \quad 60$

$x(10) = \quad 45 + \int_0^{10} v(t)\,dt =$

$45 + 85 = 130$

*Integration by Parts

(used when taking an integral of a product and the products have nothing to do with each other)

Always pick the function whose derivative goes away to be u.

There are two special cases.

Case 1: When $\ln x$ is in the problem it must be u.

Case 2: When neither equation goes away, either equation can be u (the equation we pick as u must be u both times) and we perform int. by parts twice and add to other side.

$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,du$ more simply $\int u\,dv = uv - \int v\,du$

EX#1: $\int xe^x\,dx = x\cdot e^x - \int e^x dx$

$= xe^x - e^x + C$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

Case 1: **EX#2:** $\int x^2 \ln x\,dx = \frac{x^3}{3}\cdot \ln x - \int \frac{x^2}{3}\,dx$

$= \frac{x^3}{3}\cdot \ln x - \frac{x^3}{9} + C$

$u = \ln x \qquad dv = x^2 dx$

$du = \frac{1}{x} \qquad v = \frac{x^3}{3}$

*Tabular method

EX#1: $\int x^2 \cos x\,dx = x^2 \sin x + 2x\cos x - 2\sin x + C$

Deriv.		Integral
x^2	$+$	$\cos x$
$2x$	$-$	$\sin x$
2	$+$	$-\cos x$
		$-\sin x$

EX#2: $\int x\cdot 3^x\,dx = \frac{x\cdot 3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2} + C$

Deriv.		Integral
x	$+$	3^x
1	$-$	$\frac{3^x}{\ln 3}$
0		$\frac{3^x}{(\ln 3)^2}$

*Special case 2

(neither function's derivative goes away so we use integration by parts twice and add integral to the other side)

1st time

$u = e^x \qquad dv = \sin x$

$du = e^x dx \qquad v = -\cos x$

2nd time

$u = e^x \qquad dv = \cos x$

$du = e^x dx \qquad v = \sin x$

EX#1: $\int e^x \sin x\,dx = -e^x + \int e^x \cos x\,dx$

$= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$

$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x$

$\int e^x \sin x\,dx = \frac{-e^x \cos x + e^x \sin x}{2} + C$