Calculus Name______________________

Avg Rate of Change, Instant Rate of Change, Def of Deriv

For each problem, find the average rate of change of the function over the given interval and also, using the definition of the derivative, find the instantaneous rate of change at the leftmost value of the given interval. Show all work!

1) $y = 2x^2 - 1$; $[-2, -1]$

Avg: $\left.\begin{matrix} x=-2;\ y=7 \\ x=-1;\ y=1 \end{matrix}\right\}\ \dfrac{7-1}{-2+1} = \boxed{-6}$

Instant: $\lim\limits_{h\to 0} \dfrac{2(x+h)^2-1-(2x^2-1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2x^2+4xh+2h^2-1-2x^2+1}{h}$

$= \lim\limits_{h\to 0} \dfrac{4xh+2h^2}{h} = \lim\limits_{h\to 0}(4x+2h) = \boxed{4x} = \boxed{-8}$

$x=-2$

2) $y = 2x^2 + 1$; $[-1, 0]$

Avg: $\left.\begin{matrix} x=-1;\ y=3 \\ x=0;\ y=1 \end{matrix}\right\}\ \dfrac{3-1}{-1} = \boxed{-2}$

Instant: $\lim\limits_{h\to 0} \dfrac{2(x+h)^2+1-(2x^2+1)}{h}$

$= \lim\limits_{h\to 0} \dfrac{2x^2+4xh+2h^2+1-2x^2-1}{h}$

$= \lim\limits_{h\to 0} \dfrac{4xh+2h^2}{h} = \lim\limits_{h\to 0}(4x+2h) = \boxed{4x} = \boxed{-4}$

$x=-1$

3) $y = -x^2 + x - 2$; $[0, 1]$

Avg: $\left.\begin{matrix} x=0;\ y=-2 \\ x=1;\ y=-2 \end{matrix}\right\}\ \dfrac{-2+2}{-1} = \boxed{0}$

Instant: $\lim\limits_{h\to 0} \left(\dfrac{-(x+h)^2+(x+h)-2-(-x^2+x-2)}{h}\right)$

$= \lim\limits_{h\to 0} \dfrac{-x^2-2xh-h^2+x+h-2+x^2-x+2}{h}$

$= \lim\limits_{h\to 0} \dfrac{-2xh-h^2+h}{h} = \lim\limits_{h\to 0}(-2x-h+1) = \boxed{-2x+1}$

at $x=0$: $\boxed{1}$

4) $y = -x^2 - x - 2$; $[-2, -\frac{3}{2}]$

Avg: $\left.\begin{matrix} x=-2;\ y=-4+2-2=-4 \\ x=-\frac{3}{2};\ y=-\frac{9}{4}+\frac{3}{2}-2=\frac{-11}{4} \end{matrix}\right\}\ \dfrac{-4+\frac{11}{4}}{-2+\frac{3}{2}} = \dfrac{-\frac{5}{4}}{-\frac{1}{2}} = \boxed{\frac{5}{2}}$

Instant: $\lim\limits_{h\to 0} \left(\dfrac{-(x+h)^2-(x+h)-2-(-x^2-x-2)}{h}\right)$

$= \lim\limits_{h\to 0} \left(\dfrac{-x^2-2xh-h^2-x-h-2+x^2+x+2}{h}\right)$

$= \lim\limits_{h\to 0} \left(\dfrac{-2xh-h^2-h}{h}\right) = \lim\limits_{h\to 0}(-2x+h-1) = \boxed{-2x-1}$

at $x=-2$: $\boxed{3}$

5) $y = \dfrac{1}{x}$; $[2, 3]$

Avg: $\left.\begin{matrix} x=2;\ y=\frac{1}{2} \\ x=3;\ y=\frac{1}{3} \end{matrix}\right\}\ \dfrac{\frac{1}{2}-\frac{1}{3}}{2-3} = \boxed{-\frac{1}{6}}$

Instant: $\lim\limits_{h\to 0} \left(\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}\right)$

$= \lim\limits_{h\to 0} \left(\dfrac{\frac{x-(x+h)}{x(x+h)}}{h}\right) = \lim\limits_{h\to 0} \left(\dfrac{\frac{-h}{x(x+h)}}{h}\right) = \lim\limits_{h\to 0} \left(\dfrac{-1}{x(x+h)}\right)$

$= \boxed{\dfrac{-1}{x^2}}$

at $x=2$: $\boxed{-\frac{1}{4}}$

6) $y = -\dfrac{1}{x-3}$; $[-1, 0]$

Avg: $\left.\begin{matrix} x=-1;\ y=\frac{1}{4} \\ x=0;\ y=\frac{1}{3} \end{matrix}\right\}\ \dfrac{\frac{1}{4}-\frac{1}{3}}{-1-0} = \boxed{\frac{1}{12}}$

Instant: $\lim\limits_{h\to 0} \left(\dfrac{\frac{-1}{x+h-3}+\frac{1}{x-3}}{h}\right)$

$= \lim\limits_{h\to 0} \left(\dfrac{\frac{-x+3+x+h-3}{(x+h-3)(x-3)}}{h}\right) = \lim\limits_{h\to 0} \left(\dfrac{\frac{h}{(x+h-3)(x-3)}}{h}\right)$

$= \lim\limits_{h\to 0} \dfrac{1}{(x+h-3)(x-3)} = \boxed{\dfrac{1}{(x-3)^2}}$

at $x=-1$: $\boxed{\frac{1}{16}}$

$y - y_1 = m(x - x_1)$

For each problem, find the equation of the line tangent to the function at the given point. Your answer should be in slope-intercept form.

7) $y = -x^2 - 2x + 2$ at $(-1, 3)$

$y - y_1 = m(x - x_1)$ Need a point $(-1, 3)$ and a slope (the derivative at $x = -1$):

$$y' = \lim_{h\to 0}\left(\frac{-(x+h)^2 - 2(x+h) + 2 - (-x^2 - 2x + 2)}{h}\right)$$

$$= \lim_{h\to 0} \frac{-\cancel{x^2} - 2xh - h^2 \cancel{-2x} - 2h \cancel{+2} \cancel{+x^2} \cancel{+2x} \cancel{-2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-2x - h - 2)}{\cancel{h}} = -2x - 2\Big|_{x=-1} = 0$$

$$y - 3 = 0(x + 1) \rightarrow \boxed{y = 3}$$

8) $y = x^2 + 8x + 10$ at $(-3, -5)$

$$y' = \lim_{h\to 0}\left(\frac{(x+h)^2 + 8(x+h) + 10 - (x^2 + 8x + 10)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{x^2} + 2xh + h^2 \cancel{+8x} + 8h \cancel{+10} \cancel{-x^2} \cancel{-8x} \cancel{-10}}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + 8)}{\cancel{h}} = 2x + 8\Big|_{x=-3} = 2$$

$$y + 5 = 2(x + 3)$$

$$\boxed{y = 2x + 1}$$

9) $y = -2x^2 - 8x - 9$ at $(-3, -3)$

$$y' = \lim_{h\to 0}\left(\frac{-2(x+h)^2 - 8(x+h) - 9 - (-2x^2 - 8x - 9)}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{-2x^2} - 4xh - 2h^2 \cancel{-8x} - 8h \cancel{-9} \cancel{+2x^2} \cancel{+8x} \cancel{+9}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-4x - 2h - 8)}{\cancel{h}} = -4x - 8\Big|_{x=-3} = 4$$

$$y + 3 = 4(x + 3)$$

$$\boxed{y = 4x + 9}$$

10) $y = -x^2 + 2x + 2$ at $(2, 2)$

$$y' = \lim_{h\to 0}\left(\frac{-(x+h)^2 + 2(x+h) + 2 - (-x^2 + 2x + 2)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{-x^2} - 2xh - h^2 \cancel{+2x} + 2h \cancel{+2} \cancel{+x^2} \cancel{-2x} \cancel{-2}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{h}(-2x - h + 2)}{\cancel{h}}\right) = -2x + 2\Big|_{x=2} = -2$$

$$y - 2 = -2(x - 2)$$

$$\boxed{y = -2x + 6}$$

For each problem, find the equation of the line normal to the function at the given point. If the normal line is a vertical line, indicate so. Otherwise, your answer should be in slope-intecept form. Slope of the normal = negative reciprocal of the slope of the tangent.

11) $y = x^2 - 6x + 4$ at $(2, -4)$

$$y' = \lim_{h\to 0}\left(\frac{(x+h)^2 - 6(x+h) + 4 - (x^2 - 6x + 4)}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{-6x} - 6h \cancel{+4} \cancel{-x^2} \cancel{+6x} \cancel{-4}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h - 6)}{\cancel{h}} = 2x - 6\Big|_{x=2} = -2$$

Therefore, the slope of the normal $= \frac{1}{2}$:

$$y + 4 = \tfrac{1}{2}(x - 2)$$

$$\boxed{y = \tfrac{1}{2}x - 5}$$

12) $y = x^2 + 8x + 14$ at $(-1, 7)$

$$y' = \lim_{h\to 0}\left(\frac{(x+h)^2 + 8(x+h) + 14 - (x^2 + 8x + 14)}{h}\right)$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{+8x} + 8h \cancel{+14} \cancel{-x^2} \cancel{-8x} \cancel{-14}}{h}$$

$$= \lim_{h\to 0} \frac{h(2x + h + 8)}{h} = 2x + 8\Big|_{x=-1} = 6$$

Therefore, the slope of the normal $= -\frac{1}{6}$

$$y - 7 = -\tfrac{1}{6}(x + 1)$$

$$\boxed{y = -\tfrac{1}{6}x + \tfrac{41}{6}}$$