Definite Integral

Given a function $f(x)$ that is continuous on the interval [a,b] we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

Properties

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. We can interchange the limits on any definite integral, all that we need to do is tack a minus sign onto the integral when we do.
2. $\int_a^a f(x)\,dx = 0$. If the upper and lower limits are the same then there is no work to do, the integral is zero.
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any number. So, as with limits, derivatives, and indefinite integrals we can factor out a constant.
4. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. We can break up definite integrals across a sum or difference.
5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where c is any number. This property is more important than we might realize at first. One of the main uses of this property is to tell us how we can integrate a function over the adjacent intervals, [a,c] and [c,b]. Note however that c doesn't need to be between a and b.
6. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$. The point of this property is to notice that as long as the function and limits are the same the variable of integration that we use in the definite integral won't affect the answer.

More Properties

7. $\int_a^b c\,dx = c(b-a)$, c is any number.
8. If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$.
9. If $f(x) \ge g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.
10. If $m \le f(x) \le M$ for $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

11. $$\left|\int_a^b f(x)dx\right| \le \int_a^b \left|f(x)\right|dx$$ $\left|\int_a^b f(x)dx\right| \le \int_a^b \left|f(x)\right|dx$

Fundamental Theorem of Calculus, Part I

If $f(x)$ $f(x)$ is continuous on [a,b] then,

$$g(x) = \int_a^x f(t)\,dt$$

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is continuous on [a,b] and it is differentiable on (a,b) (a,b) and that,

$$g'(x) = f(x)$$

$g'(x) = f(x)$

Fundamental Theorem of Calculus, Part II

Suppose $f(x)$ $f(x)$ is a continuous function on [a,b] and also suppose that $F(x)$ $F(x)$ is any anti-derivative for $f(x)$ $f(x)$. Then,

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

Average Function Value

The average value of a function $f(x)$ $f(x)$ over the interval [a,b] is given by,

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

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The Mean Value Theorem for Integrals

If $f(x)$ $f(x)$ is a continuous function on [a,b] then there is a number c in [a,b] such that,

$$\int_a^b f(x)dx = f(c)(b-a)$$

$\int_a^b f(x)dx = f(c)(b-a)$

In the first case we will use,

$$A = \int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right)dx, \qquad a \le x \le b$$

$A = \int_a^b \left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right) - \left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right)dx, \qquad a \le x \le b$

(3)

In the second case we will use,

$$A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy, \qquad c \le y \le d$$

$$A = \int_c^d \left(\begin{array}{c} \text{right} \\ \text{function} \end{array} \right) - \left(\begin{array}{c} \text{left} \\ \text{function} \end{array} \right) dy, \qquad c \le y \le d$$

(4)

In the final the Area and Volume Formulas section of the Extras chapter we derived the following formulas for the volume of this solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

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In the case that we get a ring the area is,

$$A = \pi \left(\left(\begin{array}{c} \text{outer} \\ \text{radius} \end{array} \right)^2 - \left(\begin{array}{c} \text{inner} \\ \text{radius} \end{array} \right)^2 \right)$$

$$A = \pi \left(\left(\begin{array}{c} \text{outer} \\ \text{radius} \end{array} \right)^2 - \left(\begin{array}{c} \text{inner} \\ \text{radius} \end{array} \right)^2 \right)$$

Volumes of Solids of Revolution / Method of Cylinders

In the previous section we started looking at finding volumes of solids of revolution. In that section we took cross sections that were rings or disks, found the cross-sectional area and then used the following formulas to find the volume of the solid.

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

$$V = \int_a^b A(x)\,dx \qquad\qquad V = \int_c^d A(y)\,dy$$

In the previous section we only used cross sections that were in the shape of a disk or a ring. This however does not always need to be the case. We can use any shape for the cross sections as long as it can be expanded or contracted to completely cover the solid we're looking at. This is a good thing because as our first example will show us we can't always use rings/disks.

The method used in the last example is called the **method of cylinders** or **method of shells**. The formula for the area in all cases will be,

$$A = 2\pi(\text{radius})(\text{height})$$

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